Linear Differential Operators in the Sciences

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Nonlinear Algebra Seminar Online
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Outline

1. Algebraic Analysis
   - some basics about the Weyl algebra D
   - holonomic functions
2. Applications
   - numerical evaluation scheme
   - statistical inference of data
   - particle physics

Algebraic analysis

The **Weyl algebra** is the free algebra over $\mathbb{C}$

$$D := \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$$

modulo the following relations. All generators commute, except ∂_i and x_i:

$$[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1 \quad \text{for} \quad i = 1, \ldots, n.$$

Some facts

- D gathers linear differential operators with polynomial coefficients

$$P = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \; c_\alpha \in \mathbb{C}[x_1, \ldots, x_n] \quad \rightsquigarrow \quad P \bullet f = 0$$

- left D-ideals encode systems of linear PDEs
- more generally: sheaves of $\mathcal{D}_X$-modules on a smooth algebraic variety X

Holonomic functions

Principal symbols

The **characteristic ideal** of a D-ideal I is

$$\mathrm{in}_{(\mathbf{0},\mathbf{1})}(I) \coloneqq \langle \mathrm{in}_{(\mathbf{0},\mathbf{1})}(P) \mid P \in I \rangle \ \triangleleft\ \mathbb{C}[x_1,\ldots,x_n,\partial_1,\ldots,\partial_n].$$

Holonomicity

A D-ideal I is **holonomic** if $\dim(\mathrm{in}_{(\mathbf{0},\mathbf{1})}(I)) = n$. A function f is **holonomic** if its annihilating D-ideal is holonomic.

Theorem (Cauchy–Kovalevskaya)

Let I be a D-ideal. Outside the singular locus of I, the $\mathbb{C}$-vector space of holomorphic solutions to I on a simply connected domain has dimension

$$\dim_{\mathbb{C}(x_1,\ldots,x_n)}(\mathbb{C}(x_1,\ldots,x_n)\langle\partial_1,\ldots,\partial_n\rangle/\mathbb{C}(x_1,\ldots,x_n)\langle\partial_1,\ldots,\partial_n\rangle\cdot I).$$

Computing with holonomic functions

Encoding

Holonomic functions are encoded by finite data (D-ideal + initial conditions)!

Holonomic gradient method [NNN+11][1]

- numerical evaluation of holonomic functions
- keeping track of the gradient by the Pfaffian system

$$\partial \bullet \left(f, f', \ldots, f^{(k-1)}\right)^{\mathrm{t}} = M \cdot \left(f, f', \ldots, f^{(k-1)}\right)^{\mathrm{t}},$$

  with $M \in \mathrm{Mat}_{k\times k}(\mathbb{C}(x))$
- holonomic gradient descent: minimization method based on the HGM

[1] Nakayama–Nishiyama–Noro–Ohara–Sei–Takayama–Takemura

HGM in statistics

- applied to the Fisher model for sampling data from SO(3) in (Sei–Shibata–Takemura–Ohara–Takayama, 2013)
  - D-ideal for SO(n) studied in (Koyama, 2020)
  - generalization to compact Lie groups other than SO(n) in (Adamer–Lőrincz–S.–Sturmfels, 2020)

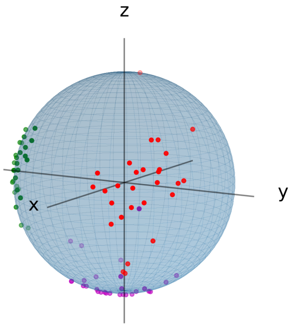

Figure: Figure 1 in (Adamer–Lőrincz–S.–Sturmfels, 2020): a dataset from a study in vectorcardiography (Downs–Liebman–Mackay, 1974)

Algebraic Analysis of ${}_1F_1$ of a Matrix Argument

- HGM applied to the cumulative distribution function of the largest eigenvalue of a Wishart matrix in (Hashigushi–Numata–Takayama–Takemura, 2013)
- closely related to the hypergeometric function ${}_1F_1$ of a matrix argument
  - construction of an annihilating D-ideal in (Muirhead, 1982)

D-module-intrinsic description of ${}_pF_q$ of a matrix argument

Not known! (except for some particular values of p, q)

Theorem (Görlach–Lehn–S., 2021)

The singular locus of Muirhead's D-ideal for ${}_1F_1$ is the hyperplane arrangement

$$\mathcal{A} := \left\{ x \in \mathbb{C}^m \mid \prod_{i=1}^{m} x_i \cdot \prod_{k \neq \ell} (x_k - x_\ell) = 0 \right\}.$$

Flipping a biased coin twice

Discrete statistical experiment: Flip a biased coin. If it shows *head*, flip again.

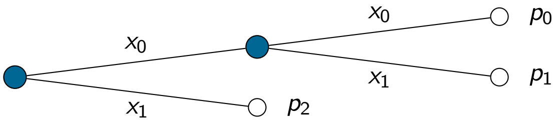

Figure: Staged tree modeling the experiment (Collazo–Görgen–Smith, 2018)

- (s_0, s_1, s_2) count of outcome when repeating this experiment
- maximum likelihood estimate:

$$\Psi(s_0, s_1, s_2) = \left(\frac{(2s_0 + s_1)^2}{(2s_0 + 2s_1 + s_2)^2}, \frac{(2s_0 + s_1)(s_1 + s_2)}{(2s_0 + 2s_1 + s_2)^2}, \frac{s_1 + s_2}{2s_0 + 2s_1 + s_2} \right)$$

Bernstein–Sato ideals

$F = (f_1, \ldots, f_p) \in \mathbb{C}[x_1, \ldots, x_n]^p$ a tuple of polynomials

Definition

The **Bernstein–Sato ideal** of F is the ideal B_F in $\mathbb{C}[s_1, \ldots, s_p]$ consisting of polynomials b for which there exists $P \in D[s_1, \ldots, s_p]$ such that

$$P \bullet \left(f_1^{s_1+1} \cdots f_p^{s_p+1} \right) = b \cdot f_1^{s_1} \cdots f_p^{s_p}.$$

Example: $F = (x^2, x(1-x), 1-x)$

Computed with the library `dmod_lib` in `Singular`:

$$B_F = \langle \prod_{k=1}^{3} (2s_0 + s_1 + k) \cdot \prod_{\ell=1}^{2} (s_1 + s_2 + \ell) \rangle \triangleleft \mathbb{C}[s_0, s_1, s_2].$$

Rigorous explanation in (S.–van der Veer, 2022)

- linking Bernstein–Sato theory, likelihood geometry, and tropical geometry

An integral transform

Definition

The **Mellin transform** of a function $f\colon \mathbb{R}^n_{>0} \to \mathbb{C}$ is

$$\mathfrak{M}\{f\}(\nu) \coloneqq \int_{\mathbb{R}^n_{>0}} x_1^{\nu_1 - 1} \cdots x_n^{\nu_n - 1} f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n.$$

One reads:

$$\mathfrak{M}\{x_i \cdot f\}(\nu) = \mathfrak{M}\{f\}(\nu + e_i) \quad \text{and} \quad \mathfrak{M}\{x_i\partial_i \bullet f\}(\nu) = -\nu_i \cdot \mathfrak{M}\{f\}(\nu).$$

Algebraic translation

- $\mathfrak{M}\{\cdot\}$ is an isomorphism of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\langle \partial_1, \ldots, \partial_n \rangle$ and the **shift algebra** $S_n \coloneqq \mathbb{C}[\nu_1, \ldots, \nu_n]\langle \sigma_1^{\pm 1}, \ldots, \sigma_n^{\pm 1} \rangle$ via $x_i^{\pm 1} \mapsto \sigma_i^{\pm 1}$, $x_i\partial_i \mapsto -\nu_i$.
- Annihilating differential operators of f translate to shift-relations of $\mathfrak{M}\{f\}$.

Particle physics

G	a Feynman graph
$\mathcal{G}$	sum of the two Symanzik polynomials of G
d	dimension of Minkowski space

Lee–Pomeransky representation of Feynman integrals

Up to some normalization, the **Feynman integral** of G is

$$I_G = \mathfrak{M}\{\mathcal{G}^{-d/2}\}.$$

Shift-relations among Feynman integrals

- ◇ Through $\mathfrak{M}\{\cdot\}$, $\mathrm{Ann}_{D_n[s]}(f^s)$ translates to $\mathrm{Ann}_{S_n[s]}(\mathfrak{M}\{f^s\})$.
- ◇ The Bernstein–Sato functional equation for $f = \mathcal{G}$, $s = -d/2$, yields shift-relations among I_G in dimensions d and $d-2$.

Thank you very much for your attention!

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