An Algebraic Invariant of Multiparameter Persistence Modules

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Topological Data Analysis

Studying the shape of data...

...with tools from (algebraic) topology

Applications

- medical and life sciences
- distinguishing point processes on a unit square
- topological machine learning

...whenever data arise!

Behind the scenes

- commutative algebra
- algebraic geometry

New invariants of multigraded modules...

...arising from TDA

Barcoding

Main tool: persistent homology

Associating barcodes to data

Input: point cloud $\{p_i\} \subseteq \mathbb{R}^N$

1. $X_\varepsilon := \cup_{p_i} B_\varepsilon(p_i)$
2. increase $\varepsilon \overset{\text{nerve}}{\rightsquigarrow}$ filtered simplicial complex
3. for all n: n-th homology with coefficients in $\mathbb{K}$ naturally is a finitely generated $\mathbb{N}$-graded module P_n over $\mathbb{K}[x]$
4. structure theorem for finitely generated modules over PIDs:

$$P_n \cong \bigoplus_i \mathbb{K}[x]x^{\alpha_i} \oplus \bigoplus_j \mathbb{K}[x]x^{\beta_j}/\mathbb{K}[x]x^{\beta_j+\gamma_j}$$

Output: barcode $\{[\alpha_i, \infty),\ [\beta_j, \beta_j + \gamma_j)\}$

Fact: This invariant is discrete, complete, and stable.

Multiparameter persistence

Study of **multifiltered** simplicial complexes (Carlsson–Zomorodian, 2009)

Algebraic counterpart

$\mathbb{N}^r$-graded $\mathbb{K}[x_1, \ldots, x_r]$-modules $M = \oplus_{a \in \mathbb{N}^r} M_a$ $\quad \deg(x_i) = e_i \in \mathbb{N}^r$

Challenges

- no higher-dimensional analogue of barcodes
- lack of **stable**, **algorithmic** invariants

Multipersistence modules as functors

Let $G \in \{\mathbb{N}^r, \mathbb{R}^r_{\geq 0}\}$ (more general monoids in (Corbet–Kerber, 2018))

$$\begin{array}{ccc} \mathrm{Fun}((G,\leq),\mathrm{Vect}_{\mathbb{K}}) & \overset{\text{isom. of cats}}{\cong} & G\text{-graded } \mathbb{K}[G]\text{-modules} \\ \cup & & \cup \\ \mathrm{Tame}((G,\leq),\mathrm{Vect}_{\mathbb{K}}) & \cong & \text{finitely presented } G\text{-graded } \mathbb{K}[G]\text{-modules} \end{array}$$

Turning discrete into stable invariants

T	a set
f	a discrete invariant $f\colon T \to \mathbb{N}$
d	an extended pseudometric $d\colon T \times T \to \mathbb{R}_{\geq 0} \cup \{\infty\}$
$\mathcal{M}$	measurable functions $[0,\infty) \to [0,\infty)$ endowed with interleaving distance

Definition & Theorem (Gäfvert–Chachólski, 2017)

The **hierarchical stabilization** of f at $x \in T$, denoted $\widehat{f}(x) \in \mathcal{M}$, is

$$\widehat{f}(x)(\tau) \;:=\; \min\{f(y) \mid y \in T\colon d(x,y) \leq \tau\}.$$

For any choice of d, $\widehat{f}\colon T \to \mathcal{M}$ is 1-Lipschitz.

Measuring distances between tame functors

How to construct metrics—in best case in a way that is suitable for learning tasks?

Pseudometrics arising from contours

$\mathbf{R}^r_\infty$ the poset obtained from adding one element ∞ to $\mathbb{R}^r_{\geq 0}$

Definition

A **persistence contour** is a functor $C\colon \mathbf{R}^r_\infty \times \mathbb{R}_{\geq 0} \to \mathbf{R}^r_\infty$ such that for every $x \in \mathbf{R}^r_\infty$, $\tau, \varepsilon \in \mathbb{R}_{\geq 0}$:

1. $x \leq C(x, \varepsilon)$ and
2. $C(C(x, \varepsilon), \tau) \leq C(x, \varepsilon + \tau)$.

ε-neighborhoods of 0

For $\varepsilon \in \mathbb{R}_{\geq 0}$ define

$$\mathcal{D}_\varepsilon \;:=\; \{G \in \mathrm{Tame}(\mathbb{R}^r_{\geq 0}, \mathrm{Vect}_{\mathbb{K}}) \mid C(x, \varepsilon) \neq \infty \;\Rightarrow\; G(x \leq C(x, \varepsilon)) = 0\}.$$

Some contours ($r = 1$)

A persistence contour is a functor $C\colon [0,\infty] \times [0,\infty) \to [0,\infty]$ such that

1. $x \leq C(x,\varepsilon)$ and
2. $C(C(x,\varepsilon),\tau) \leq C(x,\varepsilon+\tau)$.

Standard contour

$C(x,\varepsilon) \coloneqq x + \varepsilon$ is the **standard contour**.

Distance contour

Let $f\colon [0,\infty) \to [0,\infty)$ be non-decreasing such that $f(0) \geq 1$. Then there is a unique $D_f(x,\varepsilon)$ in $[x,\infty)$ for which $\varepsilon = \int_x^{D_f(x,\varepsilon)} f(y)\,\mathrm{d}y$. D_f is **of distance type**.

Shift contour

For $x \in [0,\infty)$, there is a unique $C \in [0,\infty)$ such that $x = \int_0^C f(y)\,\mathrm{d}y$. The contour $S_f(x,\varepsilon) \coloneqq \int_0^{C+\varepsilon} f(y)\,\mathrm{d}y$ is **of shift type**.

Constructing pseudometrics from contours

$C\colon \mathbf{R}^r_\infty \times [0,\infty) \to \mathbf{R}^r_\infty$ a persistence contour
$V, W \in \mathrm{Tame}(\mathbf{R}^r, \mathrm{Vect}_{\mathbb{K}})$ tame persistence modules

1. A map $f\colon V \to W$ is an ε-**equivalence** if for every $x \in \mathbf{R}^r_\infty$ such that $C(x,\varepsilon) < \infty$, there exists a linear function $W_x \to V_{C(x,\varepsilon)}$ making the following diagram commute:

$$\begin{array}{ccc} V_x & \xrightarrow{V_{x\le C(x,\varepsilon)}} & V_{C(x,\varepsilon)} \\ {\scriptstyle f_x}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle f_{C(x,\varepsilon)}} \\ W_x & \xrightarrow[V_{x\le C(x,\varepsilon)}]{} & W_{C(x,\varepsilon)} \end{array}$$

2. V and W are ε-**equivalent** if there exists a tame functor X and maps $V \xrightarrow{f} X \xleftarrow{g} W$ s.t. f is an ε_1-equivalence, g is an ε_2-equivalence, and $\varepsilon_1 + \varepsilon_2 \le \varepsilon$.
3. Let $S := \{\varepsilon \in [0,\infty) \mid V \text{ and } W \text{ are } \varepsilon\text{-equivalent}\}$. Define

$$d_C(V,W) := \begin{cases} \infty & \text{if } S \text{ is empty,} \\ \inf(S) & \text{otherwise.} \end{cases}$$

Then d_C is a pseudometric on $\mathrm{Tame}\left(\mathbb{R}^r_{\ge 0}, \mathrm{Vect}_{\mathbb{K}}\right)$.

Point processes

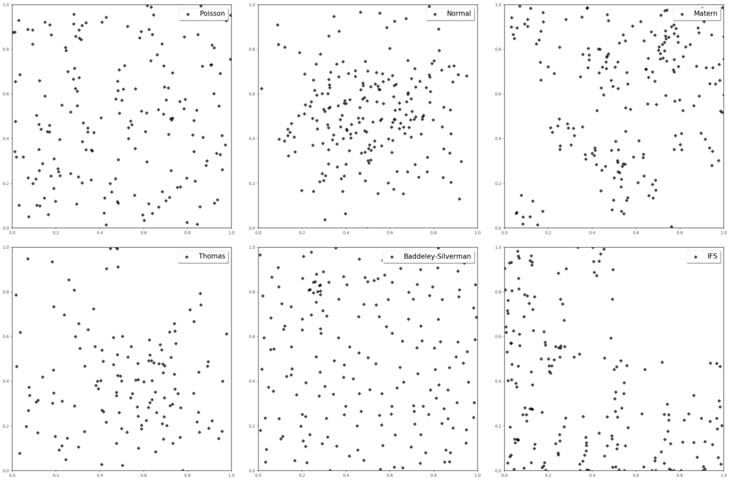

Figure: Example realizations of point processes on the unit square[1]

[1] Figure 3 in (Chachólski–Riihimäki, 2020)

Distinguishing point processes via contours

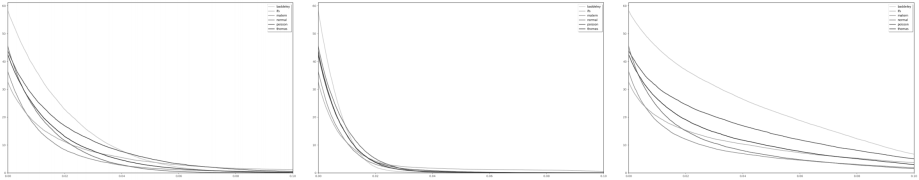

Figure: Mean H_1 stable rank of 200 simulations of point processes with respect to the standard contour (left), distance contour (middle), and shift contour (right)[3]

[3]Figure 5 in (Chachólski–Riihimäki, 2020)

Multigraded Betti numbers

Hilbert's syzygy theorem

Every f.g. $\mathbb{N}^r$-graded $\mathbb{K}[x_1, \ldots, x_r]$-module M has a minimal free resolution $F_\bullet$ of length at most r, i.e., there exists an exact sequence of $\mathbb{N}^r$-graded modules

$$F_\bullet\colon\ F_r \xrightarrow{\delta_r} \cdots \longrightarrow F_0 \xrightarrow{\delta_0} M \longrightarrow 0,$$

such that the ranks of the F_i are simultaneously minimized.

Definition

The rank of F_i in a minimal free resolution of M as above is called the **i-th total multigraded Betti number** of M and is denoted by $\beta_i(M)$.

Computing $\widehat{\beta_0}$ ...

- ... is NP-hard in general (Gäfvert–Chachólski, 2017)
- linear-time algorithm for quotients of monomial ideals in the bivariate case (Chachólski–Corbet–S., 2021)

An invariant of multigraded modules

M a finitely generated $\mathbb{N}^r$-graded $\mathbb{K}[x_1, \ldots, x_r]$-module

Theorem & Definition (Chachólski–Corbet–S., 2021)

The hierarchical stabilization of β_0 w.r.t. the metric arising from the standard contour in the direction of $v = (v_1, \ldots, v_r) \in \mathbb{N}^r$ gives rise to

$$\dim_v(M) = \min \{ \ell \mid \exists m_1, \ldots, m_\ell \in M \colon x_1^{v_1} \cdots x_r^{v_r} \cdot M \subseteq \langle m_1, \ldots, m_\ell \rangle \},$$

the **shift-dimension** of M. Such $\{m_1, \ldots, m_\ell\}$ **v-generate** M and are a **v-basis** of M for $\ell = \dim_v(M)$.

To be, or not to be in a v-basis, that is the question.

Lemma

An element $m \in M$ can be extended to a v-basis of M if and only if

$$\dim_v(M/\langle m\rangle) = \dim_v(M) - 1.$$

Proof.

- $\Rightarrow$ If $\{m, m_2, \ldots, m_{\dim_v(M)}\}$ is a v-basis of M, then $\{m_2, \ldots, m_{\dim_v(M)}\}$ is a v-basis of $M/\langle m\rangle$.
- $\Leftarrow$ If $\{[m_2], \ldots, [m_{\dim_v(M)}]\}$ is a v-basis of $M/\langle m\rangle$, then the elements $m, m_2, \ldots, m_{\dim_v(M)}$ v-generate M.

□

Examples of the shift-dimension

0-dimension

$\dim_0(M) = \operatorname{rank}(M)$, the minimal number of (homogeneous) generators of M

Free multigraded modules

$v = (1, \ldots, 1) \in \mathbb{N}^r$. $F = \mathbb{K}[x_1, \ldots, x_r](-a_1, \ldots, -a_r) \cong \mathbb{K}[x_1, \ldots, x_r] \cdot x_1^{a_1} \cdots x_r^{a_r}$, $a_1, \ldots, a_r \in \mathbb{N}$, is nv-generated by $x_1^{a_1+n} \cdots x_r^{a_r+n}$. Hence

$$(\dim_{nv}(F))_{n \in \mathbb{N}} = 1, 1, 1, \ldots$$

Quotient of homogeneous monomial ideals

Let $v = (1, \ldots, 1) \in \mathbb{N}^r$, $M = \langle x_1^3 x_2, x_1 x_2^3 \rangle / \langle x_1^4 x_2^4 \rangle$. Then $M, x_1 x_2 M \subseteq \langle x_1^3 x_2, x_1 x_2^3 \rangle$, $x_1^2 x_2^2 M \subseteq \langle x_1^3 x_2^3 \rangle$, and $x_1^3 x_2^3 M = 0$. Hence

$$(\dim_{nv}(M))_{n \in \mathbb{N}} = 2, 2, 1, 0, 0, \ldots$$

Visualization

$M = \langle x_1^3 x_2, x_1 x_2^3 \rangle / \langle x_1^4 x_2^4 \rangle \in \mathrm{Mod}(\mathbb{K}[x_1, x_2])$, $v = (1,1) \in \mathbb{N}^2$

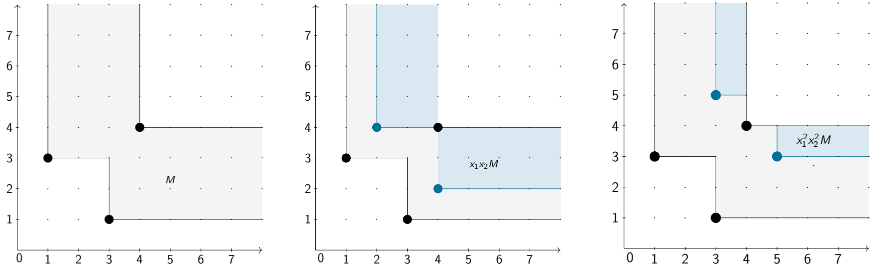

Figure: Visualization of M, $x_1 x_2 M$, and $x_1^2 x_2^2 M$

One reads:

- ◇ M, $x_1 x_2 M \subseteq \langle x_1^3 x_2, x_1 x_2^3 \rangle$, $x_1^2 x_2^2 M \subseteq \langle x_1^3 x_2^3 \rangle$, and $x_1^3 x_2^3 M = 0$.
- ◇ $\dim_{(0,0)}(M) = \dim_{(1,1)}(M) = 2$, $\dim_{(2,2)}(M) = 1$, and $\dim_{(3,3)}(M) = 0$.

Algebraic properties of the shift-dimension

Epimorphisms

If $\varphi \colon M \twoheadrightarrow N$, then $\dim_v(M) \geq \dim_v(N)$.

Proof: The image of a v-basis of M v-generates N.

Successively killing non-v-basis-elements

$m_1 \in M$ not in any v-basis of M, $[m_2]$ not in any v-basis of $M/\langle m_1 \rangle$, ...

$$M \longrightarrow M/\langle m_1 \rangle \longrightarrow \cdots \longrightarrow M/\langle m_1, \ldots, m_\ell \rangle =: M_\ell.$$

Iterating this process stabilizes after a finite number ℓ of iterations. In M_ℓ, every element is contained in some v-basis of M_ℓ.

Short exact sequences

Let $0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$ be a short exact sequence of persistence modules. Then for all $v, w \in \mathbb{N}^r$, the following two inequalities hold:

1. $\dim_{v+w}(L) \leq \dim_v(M) + \dim_w(N)$,
2. $\dim_v(L) \leq \dim_v(N) + \beta_0(M)$.

Algebraic properties of the shift-dimension

Non-additivity

In general, $\dim_v(M \oplus N) \neq \dim_v(M) + \dim_v(N)$.

Counterexample

Let $M = \langle x_1 \rangle / \langle x_1 x_2^2 \rangle$, $N = \langle x_2 \rangle / \langle x_1^2 x_2 \rangle$. Then $\dim_{(1,1)}(M) = \dim_{(1,1)}(N) = 1$. Since $x_1(x_1x_2, x_1x_2) = (x_1^2x_2, 0) = x_1x_2(x_1, 0)$, $x_2(x_1x_2, x_1x_2) = (0, x_1x_2^2) = x_1x_2(0, x_2)$ in $M \oplus N$,

$$x_1x_2\,(M \oplus N) \;\subseteq\; \langle (x_1x_2,\, x_1x_2) \rangle.$$

Hence $\dim_{(1,1)}(M \oplus N) = 1 \neq 2 = \dim_{(1,1)}(M) + \dim_{(1,1)}(N)$.

Additivity for some cases

For M, N as in one of the following three cases

1. M and N free multigraded modules
2. $\dim_v(M) \leq 1$ and N free
3. M a monomial ideal, N free of rank 1

the shift-dimension is additive, i.e., $\dim_v(M \oplus N) = \dim_v(M) + \dim_v(N)$.

Last but not least

Possible follow-up questions

- extension of our algorithm
- application to data: which information does the shift-dimension reveal?
- construction of further multipersistence contours
- stabilization of invariants other than β_0
- stability of additive version?

Thanks for your attention!

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