D-MODULES & GKZ SYSTEMS

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1. D-MODULES

Definition. $D_n := \mathbb{C}[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_n\rangle$

The (n-th) <u>Weyl algebra</u> is the non-commutative $\mathbb{C}$-algebra obtained as the free $\mathbb{C}$-algebra generated by $x_1,\dots,x_n,\partial_1,\dots,\partial_n$ modulo the following relations: all generators are assumed to commute, except ∂_i and x_i:

$$[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1 \qquad i=1,\dots,n \quad \text{(Leibniz' rule)}$$

not $\frac{\partial x_i}{\partial x_i} = 1 \ (= \partial_i \bullet x_i)$

$[\partial_i^k, x_i] = k\cdot\partial^{k-1}$

$[\partial_i, x_i^k] = k\cdot x_i^{k-1}$

An algebraic version of linear PDEs.

$P \in D$	$P \bullet f = 0$ PDE
left ideal $I \subset D$	system of PDEs
left D-module M	generalization of systems of PDEs (or: function spaces)

Theorems (Stafford):

- For every D_n-ideal I, there exist $P, Q \in D$: $I = \langle P, Q \rangle$.
- For every holonomic D-module, there exists $I \subset D$ s.t. $M \cong D/I$.

→ constructive proofs in Leykin, 2018

For $P = \sum_{a,b \in \mathbb{N}^n} c_{ab} x^a \partial^b$, let $m := \max \{ b_1 + \dots + b_n \mid c_{ab} \neq 0 \}$.

The <u>initial form</u> of P is

$$\mathrm{in}_{(0,1)}(P) := \sum_{\{b \mid \sum_{i=1}^n b_i = m\}} c_{ab} x^a \xi^b \in \underbrace{\mathbb{C}[x_1, \dots, x_n][\xi_1, \dots, \xi_n]}_{= \mathrm{gr}_{(0,1)}(D_n)}$$

$\omega = (0, \dots, 0, 1, \dots, 1) \in \mathbb{R}^{2n}$

Definitions

Let I be a left D_n-ideal.

- The <u>characteristic ideal</u> of I is the $\mathbb{C}[x_1, \dots, x_n][\xi_1, \dots, \xi_n]$-ideal

$$\mathrm{in}_{(0,1)}(I) := \langle \mathrm{in}_{(0,1)}(P) \mid P \in I \rangle \subset \mathbb{C}[x_1, \dots, x_n][\xi_1, \dots, \xi_n]$$

- I is <u>holonomic</u> if $\dim(\mathrm{in}_{(0,1)}(I)) = n$.
- The <u>holonomic rank</u> of I is $\mathrm{rank}(I) := \dim_{\mathbb{C}(x_1, \dots, x_n)} \left(R_n / R_n I \right)$

N.B.: I holonomic $\overset{\Longrightarrow}{\not\Longleftarrow}$ $\mathrm{rank}(I) < \infty$.

"Nota bene!"

Definition.

The <u>singular locus</u> of a D is the variety cut out by
$$\left(\mathrm{in}_{(0,1)}(I) : \langle \xi_1, \dots, \xi_n \rangle^\infty\right) \cap \mathbb{C}[x_1, \dots, x_n].$$
saturation + elimination

Geometrically: $\mathrm{Sing}(I) = \overline{\pi_x\left(\mathrm{Char}(I) \setminus (\mathbb{C}^n \times \{0\})\right)}$

Two different kinds of singularities: regular singularities, irregular singularities.

$n=1$: Fuchs' criterion (Newton polygon) to read if regular or irregular sing.

2. SOLUTIONS OF D-MODULES

Definition:

The <u>solution space</u> of a D_n-ideal I is the space of holomorphic solutions to I.
$$\mathrm{Sol}(I) := \{ \underbrace{f \in \mathcal{O}^{an}_{\mathbb{C}^n}(U)}_{\text{holom. fcts on } U \overset{\text{open}}{\subset} \mathbb{C}^n} \mid P \bullet f = 0 \text{ for all } P \in I \}.$$

<u>Fact:</u> $\mathrm{Sol}(I) \underset{\text{as } \mathbb{C}\text{-vs.}}{\cong} \mathrm{Hom}_{D^{an}}\left(D^{an}/D^{an}I, \mathcal{O}^{an}_{\mathbb{C}}\right)$

... on the way to the Riemann-Hilbert correspondence, a <u>positive</u> answer to Hilbert's 21st problem

Theorem (Cauchy–Kowalevskii–Kashiwara):

Let I be a holonomic D-ideal and $U \subseteq \mathbb{C}^n \setminus \mathrm{Sing}(I)$ simply connected domain. Then the space of holomorphic solutions to I has dimension $\mathrm{rank}(I)$.

↪ If I is regular singular, a basis of solutions can be computed via an algorithm of Saito–Sturmfels–Takayama based on Gröbner basis methods

Definition.

A function f is <u>holonomic</u> if its annihilating D-ideal

$$\mathrm{Ann}_D(f) := \{P \in D \mid P \bullet f = 0\}$$

is holonomic.

Why should we care about holonomic functions?

- omnipresent in applications — Feynman integrals, some probability distributions, many special functions, hypergeometric functions, ...
- can be encoded by finitely many data
- holonomic gradient method / descent
- nice closure properties (closed under $+, \cdot, *, \dots$)

RESCUE: D-algebraic fcts.

⚠ The class of holonomic functions is not closed under division or inversion.

$\sin$ is holonomic, $\frac{1}{\sin}$ is <u>not</u> holonomic

$x \mapsto xe^x$ is holonomic, but its multivalued inv. fct. (Lambert W fct) is not holonomic

3. GKZ SYSTEMS

Let $A \in \mathbb{Z}^{d \times n}$, $\beta \in \mathbb{C}^d$.

$A = (a_0 | \dots | a_n)$, $\varphi_A : (\mathbb{C}^*)^d \longrightarrow \mathbb{P}^n$,
$t \longmapsto [t^{a_0} : \dots : t^{a_n}]$

$V(\mathrm{im}(\varphi_A))$ is a toric variety

Definition.

- $I_A := \langle \partial^u - \partial^v \mid u, v \in \mathbb{N}^n : Au = Av \rangle \subset \mathbb{C}[\partial_1, \dots, \partial_n]$ the toric ideal of A
- $J_A(\beta)$ the D_n-ideal generated by the entries of $A \cdot \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} - \beta$

The D-ideal $H_A(\beta) := I_A + J_A(\beta)$ is a GKZ system (A-hypergeometric system).

Gelfand – Kapranov – Zelevinsky

$M_A(\beta) := D / H_A(\beta) \in \mathrm{Mod}(D)$

Elements of $\mathrm{Sol}(H_A(\beta))$ are called A-hypergeometric functions.

Proposition.

- For generic β, $\mathrm{rank}(H_A(\beta)) = \mathrm{vol}(A)$.
  the normalized volume of the convex hull of the origin and the columns of A
- $\mathrm{Sing}(H_A(\beta))$ is given by the principal A-determinant

Theorem: If the vector $(1, \dots, 1)$ is in the $\mathbb{Q}$-row span of A, then $H_A(\beta)$ is regular holonomic for all $\beta \in \mathbb{C}^d$. We assume so today.

Some operations on $\mathcal{D}$-modules

Let $M = D/I \in \mathrm{Mod}(D)$, $D = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$.

For $m \in \mathbb{N}$, consider $L = \{x_1 = \ldots = x_m = 0\}$ and denote $D' := \mathbb{C}[x_{m+1}, \ldots, x_n]\langle \partial_{m+1}, \ldots, \partial_n \rangle$.

Definition.

The <u>restriction of M</u> to L is the D'-module

$$N := D/(I + x_1 D + \ldots + x_m D) = D' / \underbrace{(I + x_1 D + \ldots + x_m D) \cap D'}_{\text{restriction ideal of } I}$$

Proposition.

If $f(x_1, \ldots, x_n)$ is holonomic and $f \in \mathrm{Sol}(M)$, then $f(0, \ldots, 0, x_{m+1}, \ldots, x_n)$ is a solution of N.

An integral transform.

The <u>Fourier–Laplace transform</u> is the isomorphism of non-comm. $\mathbb{C}$-algebras

$$\widehat{(\cdot)} : \mathbb{C}[x_1, \ldots, x_n]\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle \xrightarrow{\cong} \mathbb{C}[\xi_1, \ldots, \xi_n]\langle \partial_{\xi_1}, \ldots, \partial_{\xi_n} \rangle,$$

$$x_i \longmapsto -\partial_{\xi_i}, \quad \partial_i \longmapsto \xi_i.$$

Some references to dive into D-modules:

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