Exercises accompanying the lecture series "Algebraic and Holonomic Statistics"

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Exercise 1 (Another coin model).
Consider the following discrete statistical model $\mathcal{M}$ with 3 states. A gambler has a biased coin which shows *head* with probability x, and *tail* with probability $1-x$. She flips the coin thrice and records three possible outcomes: ⋄ only heads ⋄ mixed outcome ⋄ only tails

(a) Compute a parameterization and an implicitization of the model. More precisely, determine a homogeneous polynomial $f \in \mathbb{C}[p_0, p_1, p_2]$ of degree 3 which determines $\mathcal{M}$. Interpret the two images in Figure 1. Can you reproduce them in `Mathematica`?

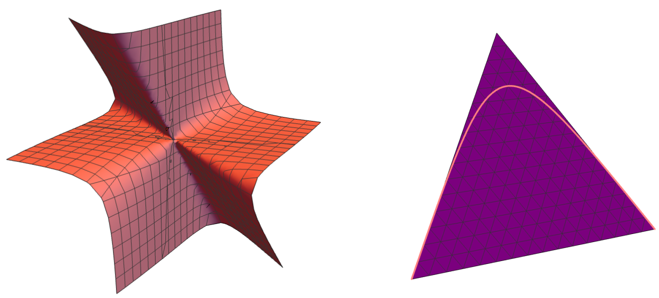

Figure 1: Images accompanying Exercise 1 (a)

(b) Determine the very affine variety X of the model and compute its ML degree $\mathrm{d}_{\mathrm{ML}}(X)$.

(c) For data $s = (s_0, s_1, s_2) \in \mathbb{N}^3$, compute the likelihood function L_s and the maximum likelihood estimator of the model in the implicitization you found in (a). To do so, you may use `Mathematica`, for instance.

(d) Compute the Bernstein–Sato ideal of the model. You can compute it using the library `Dmod_lib` in the computer algebra software `Singular`[1] by running the following code.

```
LIB "dmod.lib";
ring r = 0,x,dp; setring r;
ideal F = (x^3,3*x*(1-x),(1-x)^3);
def A = annfsBMI(F); setring A; BS;
```

In order to interpret the output of the code above, have a look at the documentation of the command `annfsBMI`: https://www.singular.uni-kl.de/Manual/4-0-3/sing_598.htm

(e) Compute the tropical variety of the model.

[1] An online version of `Singular` is available at the following link: https://www.singular.uni-kl.de:8003/

Exercise 2 (Mellin transform).
The *Mellin transform* of a complex-valued function f in n variables $x = (x_1, \dots, x_n)$ is

$$\mathfrak{M}\{f\}(\nu_1, \dots, \nu_n) \;=\; \int_\Gamma f(x_1, \dots, x_n)\, x_1^{\nu_1} \cdots x_n^{\nu_n}\, \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{x_1 \cdots x_n}\,,$$

where the integration contour Γ is such that the boundary term in IBP vanishes and the integral converges. The *(n-th) shift algebra*, denoted S_n, is the non-commutative $\mathbb{C}$-algebra obtained from the free $\mathbb{C}$-algebra that is generated by variables $\nu_1, \dots, \nu_n$ and shift-operators $\sigma_1^{\pm 1}, \dots, \sigma_n^{\pm 1}$ by imposing the following relations: all generators commute, except ν_i and $\sigma_i^{\pm 1}$. They obey the rule

$$\sigma_i \nu_i^{\pm 1} \;=\; (\nu_i \pm 1)\sigma_i^{\pm 1}\,.$$

(a) Compute $\mathfrak{M}\{x_i \cdot f\}$ and $\mathfrak{M}\{x_i \partial_i \bullet f\}$.

(b) Formulate the action of S_n on $\mathfrak{M}\{f\}$.

(c) Building on (a), formulate the Mellin transform as an isomorphism of the Weyl algebra on the torus $D_{\mathbb{G}_m^n} = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]\langle \partial_1, \dots, \partial_n \rangle$ and the shift algebra S_n. This isomorphism is referred to as "algebraic Mellin transform".

(d) Let $f = (x_1 + 1)(x_2 + 1) \in \mathbb{C}[x_1, x_2]$. Compute its s-parametric annihilator $\operatorname{Ann}_{D_2[s]}(f^s)$. In order to compute it, you can use the command `annfsBMI` in `Singular`. The s-parametric annihilator is encoded as `LD` in the output ring. Deduce shift-relations for $\mathfrak{M}\{f^s\}$.