D-Module Techniques for Solving Differential Equations in the Context of Feynman Integrals

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Motivation

Aim: exploit algebraic geometry behind Feynman integrals

- extraction of **properties** of Feynman integrals from their PDEs
- **algorithmic** computation of **series solutions** of PDEs by algebraic methods
- **evaluation** of Feynman integrals
- providing a **dictionary** between algebraic analysis and high energy physics

Outline

1. Algebraic analysis $D = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$
2. Algebraic computation of solutions $F_k(x) = x^A \cdot \sum_{p,b \text{ suitable}} c_{pb}\, x^p \log(x)^b$
3. Merging D-module and physics methods

Definition

The **Weyl algebra** is obtained from the free algebra over $\mathbb{C}$

$$D := \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$$

by imposing the following relations:

$$[\partial_i, x_j] = \partial_i x_j - x_j \partial_i = \delta_{ij} \quad \text{for} \quad i, j = 1, \ldots, n.$$

From PDEs to D-ideals and vice versa

- D gathers linear differential operators with polynomial coefficients

$$P = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha,\beta} x^\alpha \partial^\beta, \; c_{\alpha,\beta} \in \mathbb{C} \quad \rightsquigarrow \text{PDE: } \boxed{P \bullet f(x_1, \ldots, x_n) = 0}$$

Example: $P = \partial^2 - x \in D$ encodes Airy's equation $f''(x) - x \cdot f(x) = 0$.

- left D-ideals encode systems of linear PDEs
  operations with D-ideals: integral transforms, restrictions, push forward, ...

Holonomic functions

One variable

A function $f(x)$ is **holonomic** if there exists $P \in D$ that annihilates f, i.e., $P \bullet f = 0$.

Multivariate case: $f(x_1, \ldots, x_n)$ is holonomic if $\mathrm{Ann}_D(f)$ is a "holonomic" D-ideal.

Examples: Feynman integrals, hypergeometric, periods, Airy, polylogarithms, ...

Denote by $R_n = \mathbb{C}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$ the **rational Weyl algebra**.

Theorem (Cauchy–Kovalevskaya–Kashiwara)

Let I be a holonomic D-ideal. The $\mathbb{C}$-vector space of holomorphic solutions to I on a simply connected domain in $\mathbb{C}^n$ outside the singular locus of I has finite dimension

$$\mathrm{rank}(I) \;=\; \dim_{\mathbb{C}(x_1,\ldots,x_n)} (R_n/R_n I) \;.$$

$\Rightarrow$ A holonomic function is encoded by finite data!

Singularities

D-ideals can be regular singular or irregular singular.

Univariate case: read from growth behavior of general solution near singular points

Example: $\diamond$ $\log(x)$ moderate growth at $x = 0$ $\diamond$ $\exp(1/x)$ essential singularity at $x = 0$

A.-L. S. and B. Sturmfels. *D*-Modules and Holonomic Functions. Preprint arXiv:1910.01395, 2019. To appear in the volume *Varieties, polyhedra, computation* of *EMS Series of Congress Reports*.

Running example

Variables: $x_1 = |p_1|^2\,,\quad x_2 = |p_2|^2\,,\quad x_3 = |p_1 + p_2|^2.$

The D-ideal $I_3(c_0, c_1, c_2, c_3)$

Consider $I_3(c_0, c_1, c_2, c_3) = \langle P_1, P_2, P_3 \rangle \subset D_3$ arising from **conformal invariance**.
dilatations + conformal boosts

$$
\begin{aligned}
P_1 &= 4(x_1\partial_1^2 - x_3\partial_3^2) + 2(2 + c_0 - 2c_1)\partial_1 - 2(2 + c_0 - 2c_3)\partial_3\,,\\
P_2 &= 4(x_2\partial_2^2 - x_3\partial_3^2) + 2(2 + c_0 - 2c_2)\partial_2 - 2(2 + c_0 - 2c_3)\partial_3\,,\\
P_3 &= (2c_0 - c_1 - c_2 - c_3) + 2(x_3\partial_3 + x_2\partial_2 + x_1\partial_1)\,.
\end{aligned}
$$

Parameters: $c_0 = d$ spacetime dimension $\quad$ c_1, c_2, c_3 conformal weights

Choice: $I_3 := I_3(4, 2, 2, 2)$ $\widehat{=}$ conformal ϕ^4-theory in 4 spacetime dimensions

I_3 is regular singular, $\operatorname{rank}(I_3) = 4$

Remark: The D-ideal I_3 is the restriction of a GKZ system.

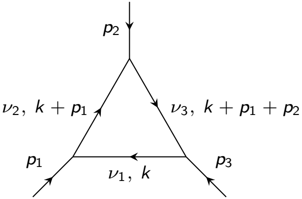
L. de la Cruz. Feynman integrals as A-hypergeometric functions. *J. High Energy Phys.*, 123(2019), 2019.

The solution space of I_3...

... is spanned by the triangle integral

$$J^{\text{triangle}}_{d;\nu_1,\nu_2,\nu_3} = \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{\mathrm{i}\pi^{\frac{d}{2}}} \frac{1}{(-|k|^2)^{\nu_1}\,(-|k+p_1|^2)^{\nu_2}\,(-|k+p_1+p_2|^2)^{\nu_3}}$$

and its analytic continuations. $\mathrm{rank}(I_3) = 4$



One-loop triangle Feynman diagram with massless propagators and massive external particles.

Unitary exponents $\nu_1 = \nu_2 = \nu_3 = 1,\ d = 4$:

$$\begin{aligned}
f_1(x_1,x_2,x_3) &= J^{\text{triangle}}_{4;1,1,1}(x_1,x_2,x_3)\,,\\
f_2(x_1,x_2,x_3) &= \frac{1}{\sqrt{\lambda}} \log\left(\frac{x_1-x_2-x_3-\sqrt{\lambda}}{x_1-x_2-x_3+\sqrt{\lambda}}\right),\\
f_3(x_1,x_2,x_3) &= \frac{1}{\sqrt{\lambda}} \log\left(\frac{x_2-x_1-x_3-\sqrt{\lambda}}{x_2-x_1-x_3+\sqrt{\lambda}}\right),\\
f_4(x_1,x_2,x_3) &= \frac{1}{\sqrt{\lambda}}\,,
\end{aligned}$$

where $\lambda = x_1^2 + x_2^2 + x_3^2 - 2(x_1x_2 + x_1x_3 + x_2x_3)$ is the **Källén** function.

Initial forms

Principal symbol ($n = 1$)

$\mathrm{in}_{(0,1)}(x\partial - x^2) = x\xi$ is the part of maximal $(0,1)$-weight $\quad \partial \rightsquigarrow \xi$

Several variables: $\mathrm{in}_{(0,1)}(x_1\partial_1 + x_2\partial_2 + 1) = x_1\xi_1 + x_2\xi_2$ in general, not a monomial

Algebraically

- The **characteristic ideal** of a D-ideal I is

$$\mathrm{in}_{(0,1)}(I) = \langle\, \mathrm{in}_{(0,1)}(P) | P \in I \,\rangle \subset \mathbb{C}[x_1, \ldots, x_n][\xi_1, \ldots, \xi_n]\,.$$

- The **characteristic variety** of I is

$$\mathrm{Char}(I) = V(\mathrm{in}_{(0,1)}(I)) = \big\{(x,\xi) \,|\, p(x,\xi) = 0 \text{ for all } p \in \mathrm{in}_{(0,1)}(I)\big\} \subset \mathbb{C}^{2n}\,.$$

- The **singular locus** $\mathrm{Sing}(I)$ of I is the vanishing set of the ideal

$$\big(\mathrm{in}_{(0,1)}(I) : \langle \xi_1, \ldots, \xi_n\rangle^{(\infty)}\big) \cap \mathbb{C}[x_1, \ldots, x_n]\,.$$ saturation + elimination

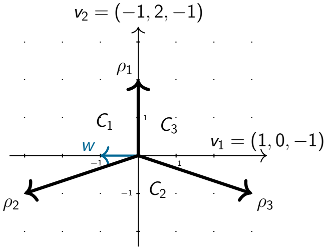
Examples

1. For $I = \langle x^2\partial + 1\rangle \subset D$, $\mathrm{in}_{(0,1)}(I) = \langle x^2\xi\rangle$ and $\mathrm{Sing}(I) = V(x) = \{0\}$. $\quad \mathbb{C}\cdot\exp(1/x)$
2. The characteristic ideal of $I = \langle x_1\partial_2, x_2\partial_1\rangle \subset D_2$ is the $\mathbb{C}[x_1, x_2, \xi_1, \xi_2]$-ideal $\langle x_1\xi_2,\, x_2\xi_1,\, x_1\xi_1 - x_2\xi_2,\, x_2\xi_2^2,\, x_2^2\xi_2\rangle$ and $\mathrm{Sing}(I) = V(x_1, x_2) \subset \mathbb{C}^2$. $\quad \mathbb{C}\cdot 1$

Gröbner deformations

Weights of the form $(-w, w)$, $w = (w_1, \dots, w_n) \in \mathbb{R}^n$

- The **w-weight** of $c_{\alpha,\beta} x^\alpha \partial^\beta$ is $-w \cdot \alpha + w \cdot \beta$.
- The **initial form** of $P = \sum c_{\alpha,\beta} x^\alpha \partial^\beta$ is the subsum of all terms of maximal w-weight.



Small Gröbner fan of $I_3 \subset D$, here drawn in $\mathbb{R}^3_w / \mathbb{R}(1,1,1)$.

Initial and indicial ideal (with respect to w)

- The **initial ideal** of I is the D-ideal $\mathrm{in}_w(I) = \langle \mathrm{in}_{(-w,w)}(P) | P \in I \rangle \subset D$.
- The **indicial ideal** of I is the $\mathbb{C}[\theta_1, \dots, \theta_n]$-ideal $\mathrm{ind}_w(I) = R_n \cdot \mathrm{in}_{(-w,w)}(I) \cap \mathbb{C}[\theta_1, \dots, \theta_n]$.

$\theta_i = x_i \partial_i$ the i-th Euler operator

The zeroes of $\mathrm{ind}_w(I)$ in $\mathbb{C}^n$ are the **exponents** of I.
The starting monomials of solutions to I will be of the form $x^A \log(x)^B$ with $A \in V(\mathrm{ind}_w(I))$.

Pipeline: from I to starting terms of series solutions

D_n-ideal I $\overset{w \in \mathbb{R}^n}{\rightsquigarrow}$ $\mathrm{in}_{(-w,w)}(I)$ $\rightsquigarrow$ $\mathrm{ind}_w(I) \subset \mathbb{C}[\theta_1, \dots, \theta_n]$ $\overset{V(\mathrm{ind}_w(I))}{\rightsquigarrow}$ $x^A \log(x)^B$

Aim: Solutions to I of the form $F_k(x) \;=\; x^A \cdot \sum_{\substack{0\le p\cdot w\le k,\, p\in C_{\mathbb{Z}}^*,\\ 0\le b_j<\mathrm{rank}(I)}} c_{pb}\, x^p \log(x)^b$.

Initial series

The **w-weight** of a monomial $x^A \log(x)^B$ is the real part of $w \cdot A$. The **initial series** $\mathrm{in}_w(f)$ of a function $f = \sum_{A,B} c_{AB} x^A \log(x)^B$ is the subsum of all terms of minimal w-weight.

Proposition

If I is regular holonomic and w a generic weight for I, there exist $\mathrm{rank}(I)$ many canonical series solutions of I which lie in the **Nilsson ring** $N_w(I)$ of I with respect to w,

$$N_w(I) \;:=\; \mathbb{C}[\![C_w(I)_{\mathbb{Z}}^*]\!][x^{e^1},\ldots,x^{e^r},\log(x_1),\ldots,\log(x_n)]\,.$$

- $C_w(I)^*$ the dual cone of the Gröbner cone of w
- $C_w(I)_{\mathbb{Z}}^* = C_w(I)^* \cap \mathbb{Z}^n$
- $\{e^1,\ldots,e^r\}$ the exponents of I

Monomial ordering $\prec_w$ refining w-weight: The number of solutions to I with starting monomial of the form $x^A \log(x)^B$ is the multiplicity of A as zero of $\mathrm{ind}_w(I)$.

M. Saito, B. Sturmfels, and N. Takayama. *Gröbner Deformations of Hypergeometric Differential Equations*, volume 6 of *Algorithms and Computation in Mathematics*. Springer, 2000.

Theorem (Saito–Sturmfels–Takayama)

Let I be a regular holonomic $\mathbb{Q}[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_n\rangle$-ideal and $w\in\mathbb{R}^n$ generic for I. Let I be given by a Gröbner basis for w. There exists an algorithm which computes all terms up to specified w-weight in the canonical series solutions to I with respect to $\prec_w$.

Procedure

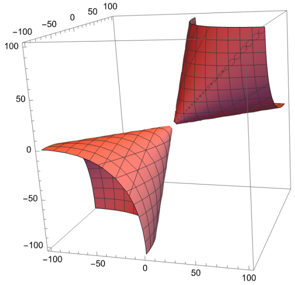
Input: A regular holonomic D_n-ideal I, its small Gröbner fan Σ in $\mathbb{R}^n$, a weight vector $w\in\mathbb{R}^n$ that is generic for I, and the desired order $k\in\mathbb{N}$.

... for each starting monomial $x^A\log(x)^B$: solving linear system modulo desired w-weight for vector spaces of monomials of same w-weight. *recurrence relations*

Output: The canonical series solutions of I with respect to w, truncated at w-weight k.

M. Saito, B. Sturmfels, and N. Takayama. *Gröbner Deformations of Hypergeometric Differential Equations*, volume 6 of *Algorithms and Computation in Mathematics*. Springer, 2000.

Starting monomials for I_3



The **singular locus** of I_3 is

$$\mathrm{Sing}\,(I_3) \;=\; V\,(x_1x_2x_3\cdot\lambda) \;\subset\; \mathbb{C}^3.$$

Vanishing locus of the Källén polynomial

$$\lambda = x_1^2 + x_2^2 + x_3^2 - 2(x_1x_2 + x_1x_3 + x_2x_3)$$

+ coordinate hyperplanes $\{x_i = 0\}$

Initial and indicial ideal for $w = (-1,0,1) \in C_1$

- $\mathrm{in}_{(-w,w)}(I_3) \;=\; \langle x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + 1,\; x_2\partial_2^2 + \partial_2,\; x_3\partial_3^2 + \partial_3\,\rangle \;\subset\; D_3$
- $\mathrm{ind}_w(I_3) \;=\; R_3\cdot \mathrm{in}_{(-w,w)}(I)\cap\mathbb{C}[\theta_1,\theta_2,\theta_3] \;=\; \langle\theta_1+\theta_2+\theta_3+1,\;\theta_2^2,\;\theta_3^2\rangle \;\subset\; \mathbb{C}\,[\theta_1,\theta_2,\theta_3]$

Exponents of I: $\boxed{V(\mathrm{ind}_w(I_3)) \;=\; \{(-1,0,0)\}\,.}$ $\quad\widehat{=}\; x_1^{-1}x_2^0x_3^0 \;=\; 1/x_1$

Change of variables: $\quad y_1 \;=\; x_1\,,\quad y_2 \;=\; x_2/x_1\,,\quad y_3 \;=\; x_3/x_1\,.$

Starting monomials of solutions *read from primary decomposition of* $\mathrm{ind}_w(I)$

- $1/y_1$
- $1/y_1\log(y_2)$
- $1/y_1\log(y_3)$
- $1/y_1\log(y_2)\log(y_3)$

Lifting the starting monomials *here displayed for f_1, f_2, f_3 for w-weight 0 to 4*

$$
\begin{aligned}
\tilde{f}_1(y_2, y_3) &= 1 + y_2 + y_3 + y_2^2 + 4y_2y_3 + y_3^2 + y_2^3 + 9y_2^2y_3 + y_2^4 + \cdots, \\
\tilde{f}_2(y_2, y_3) &= \log(y_2) + \log(y_2)y_2 + (2 + \log(y_2))y_3 + \log(y_2)y_2^2 + (4 + 4\log(y_2))y_2y_3 \\
&\quad + (3 + \log(y_2))y_3^2 + (\log(y_2))y_2^3 + (6 + 9\log(y_2))y_2^2y_3 + \log(y_2)y_2^4 + \cdots, \\
\tilde{f}_3(y_2, y_3) &= \log(y_3) + (2 + \log(y_3))y_2 + \log(y_3)y_3 + (3 + \log(y_3))y_2^2 \\
&\quad + (4 + 4\log(y_3))y_2y_3 + \log(y_3)y_3^2 + \left(\frac{11}{3} + \log(y_3)\right) y_2^3 \\
&\quad + (15 + 9\log(y_3))y_2^2y_3 + \left(\frac{25}{6} + \log(y_3)\right) y_2^4 + \cdots.
\end{aligned}
$$

Then $f_i(x_1, x_2, x_3) = 1/x_1 \cdot \tilde{f}_i(y_2, y_3)$ are canonical series solutions to I_3. (truncated)

Implementation in Sage for the bivariate case

Available at: https://mathrepo.mis.mpg.de/DModulesFeynman/

Truncation with respect to w-weight

$f(x_1, \ldots, x_n)$ general solution of a regular holonomic D-ideal I

Capturing the weight vector via an auxiliary variable

Choose a generic weight $w \in \mathbb{R}^n$ for I. Set

$$f_w(t, x_1, \ldots, x_n) := f(t^{w_1} x_1, \ldots, t^{w_n} x_n).$$

Merging with canonical series solutions

1. From I, derive a **Fuchsian system** for $f_w(t, x_1, \ldots, x_n)$.
2. Solve the system via the path-ordered exponential formalism.
3. Compute the asymptotic expansion of $f_w(t, x)$ around $t = 0$:

$$f_w(t, x) = \sum_{k \geq 0} \sum_{m=0}^{m_{\max}} c_{k,m}(x)\, t^k \log(t)^m .$$

   By construction, $c_{k,m}(x)$ has w-weight k .

4. Truncate the expansion at t^k and evaluate at $t = 1$. Nota bene: $f_w|_{t=1} \equiv f$.

1. F. Brown. Iterated Integrals in Quantum Field Theory. In 6*th Summer School on Geometric and Topological Methods for Quantum Field Theory*, pages 188–240, 2013.
2. W. Wasow. *Asymptotic expansions for ordinary differential equations*. Pure and Applied Mathematics, Vol. XIV. Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1965.

Conclusion

In a nutshell

1. D-ideals encode crucial properties of their solution functions
   e.g. Feynman integrals, arbitrary loop order, irrespective of whether polylogarithmic, etc.
2. algorithmic computation of truncated series solutions by algebraic methods
   no gauge transform required
3. evaluation of solution functions to desired w-weight
   freedom in choosing weight vector w
4. dictionary algebra–physics
   computing series solutions, Pfaffian system vs. Laporta's algorithm

Thank you for your attention!

J. Henn, E. Pratt, A.-L. S., and S. Zoia. *D*-Module Techniques for Solving Differential Equations in the Context of Feynman Integrals. Preprint arXiv:2303.11105, 2023.

The conformal group

$z = \left(z^0, z^1, \ldots, z^{d-1}\right)^\top$	vector of d-dimensional spacetime coordinates
$z_1 \cdot z_2 := z_1^\top \cdot g \cdot z_2$	$g = \mathrm{diag}(1, -1, \ldots, -1)$ the metric tensor
$p_1, \ldots, p_n$	momentum vectors

Translations	$z \longrightarrow z + \epsilon, \quad \epsilon \in \mathbb{R}^d$
(Proper) Lorentz transformations	$z \longrightarrow \Lambda \cdot z, \quad \Lambda \in SO(1, d-1)$
Dilatations	$z \longrightarrow e^\omega z, \quad \omega \in \mathbb{R}$
Conformal boosts	$z \longrightarrow \dfrac{z - \lvert z\rvert^2 \epsilon}{1 - 2\, z \cdot \epsilon + \lvert z\rvert^2 \lvert\epsilon\rvert^2}, \quad \epsilon \in \mathbb{R}^d$

Poincaré group symmetry group of Einstein's theory of special relativity
conformal group Poincaré + dilatations + conformal boosts

Invariance under...

- translations implies momentum conservation
- Lorentz transformation implies dependency on Mandelstam invariants $p_k \cdot p_\ell$ only

Generators in position space *to momentum space via Fourier transform*

- dilatations: $\mathfrak{D}_n = -\mathrm{i} \sum_{k=1}^n \left(z_k \cdot \partial_{z_k} + c_k\right)$
- conformal boosts: $\mathfrak{K}_n = \mathrm{i} \sum_{k=1}^n \left[\lvert z_k\rvert^2 \partial_{z_k} - 2\, z_k \left(z_k \cdot \partial_{z_k}\right) - 2\, c_k\, z_k\right]$

Running example: $n = 3$, momenta p_1, p_2, p_3, variables $x_i = \lvert p_i\rvert^2$

- P_3 stems from $\widehat{\mathfrak{D}_3}$
- P_1, P_2 stem from $\widehat{\mathfrak{K}_3}$

Systems in matrix form

◇ I a holonomic D_n-ideal of rank $m = \operatorname{rank}(I)$, $f \in \operatorname{Sol}(I)$
◇ $1, s_2, \ldots, s_m$ a $\mathbb{C}(x)$-basis of R_n/R_nI standard monomials for a Gröbner basis of I

Pfaffian system

Set $F = (f, s_2 \bullet f, \ldots, s_m \bullet f)^\top$. There exist $P_1, \ldots, P_n \in \mathbb{C}(x_1, \ldots, x_n)^{m \times m}$ for which

$$\boxed{\partial_i \bullet F \;=\; P_i \cdot F\,.}$$

The matrices P_i fulfill $P_iP_j - P_jP_i \;=\; \partial_i \bullet P_j - \partial_j \bullet P_i$ for all i, j. integrability

If all poles are of order at most 1, the system is **Fuchsian**. To arrive at a Fuchsian form, one might need a gauge transform. Wasow's method

Construction of a Pfaffian system	**IBP reduction with Laporta's algorithm**
∂^a	Feynman integrals
a in ∂^a	propagator powers
$\partial^a Q_i = 0$ in R_n/R_nI	IBP identities
$\mathbb{C}(x)$-basis of R_n/R_nI	set of master integrals

1. V. Chestnov, F. Gasparotto, M. K. Mandal, P. Mastrolia, S.-J. Matsubara-Heo, H. J. Munch, N. Takayama. Macaulay matrix for Feynman integrals: Linear relations and intersection numbers. *J. High Energy Phys.*, 187(2022), 2022.
2. W. Wasow. *Asymptotic expansions for ordinary differential equations*. Pure and Applied Mathematics, Vol. XIV. Interscience Publishers John Wiley & Sons, Inc., New York-London- Sydney, 1965.

The SST algorithm

Input: A regular holonomic D_n-ideal I, its small Gröbner fan Σ in $\mathbb{R}^n$, a weight vector $w \in \mathbb{R}^n$ that is generic for I, and the desired order $k+1 \in \mathbb{N}$.

1. Determine a Gröbner basis $G = \{g_1, \ldots, g_d\}$ of I with respect to w.
2. Write each $g \in G$ as $x^\alpha g = f - h$ with $\alpha \in \mathbb{Z}^n$ such that $f \in \mathbb{K}[\theta_1, \ldots, \theta_n]$ and $h \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\langle \partial_1, \ldots, \partial_n \rangle$ with $\mathrm{ord}_{(-w,w)}(h) < 0$.
3. Compute the indicial ideal $\mathrm{ind}_w(I)$ and its $\mathrm{rank}(I)$ many solutions. They are the form $x^A \log(x)^B$ with $A \in V(\mathrm{ind}_w(I))$. For each starting of these monomials, carry out Step 4.
4. Assume the partial solution

$$\boxed{F_k(x) = x^A \cdot \sum_{0 \leq p \cdot w \leq k,\, p \in C_{\mathbb{Z}}^*} c_{pb} x^p \log(x)^b .}$$

is known. Solve the linear system

$$(f_1, \ldots, f_d) \bullet E_{k+1}(x) = (h_1 - f_1, \ldots, h_d - f_d) \bullet F_k(x) \mod w\text{-weight } k+2$$

for $E_{k+1} \in \sum_{p \cdot w = k+1,\, p \in C_{\mathbb{Z}}^*} L_p'$ of w-weight $k+1$. Adding E_{k+1} to F_k lifts F_k to F_{k+1}.
L_p' the subspace of $L_p = x^A \sum_{0 \leq b_i \leq \mathrm{rank}(I)} \mathbb{K} \cdot x^p \log(x)^b$ spanned by monomials $\notin \mathrm{Start}_{\prec_w}(I)$

Output: The canonical series solutions of I with respect to w, truncated at w-weight $k+1$.

M. Saito, B. Sturmfels, and N. Takayama. *Gröbner Deformations of Hypergeometric Differential Equations*, volume 6 of *Algorithms and Computation in Mathematics*. Springer, 2000.

SST algorithm: a hypergeometric example

Consider the D-ideal I generated by $P = \theta(\theta-3) - x(\theta+a)(\theta+b)$.

1. I is holonomic of rank $\mathrm{ord}_{(0,1)}(P) = 2$.
2. Gröbner fan of I: two maximal cones $\pm\mathbb{R}_{\geq 0}$.
3. For the weight $w = 1$, $\mathrm{in}_{(-w,w)}(I) = \langle\, \theta(\theta-3) \,\rangle = \mathrm{ind}_w(I)$.
4. Exponents of I: $V(\mathrm{ind}_w(I)) = \{0, 3\}$. starting monomials $x^0 = 1$ and x^3
5. Choose x^3 as starting monomial, $L_p = \mathbb{C} \cdot \{x^{p+3}, x^{p+3}\log(x)\}$. $x^3 \sum_p c_{p,1}x^p + c_{p,2}x^p \log(x)$
6. Write $P = f - h$, where $f = \theta(\theta-3)$ and $h = x(\theta+a)(\theta+b)$. Action of θ on L_p:
$$\theta \bullet x^{p+3} = (p+3)x^{p+3} \quad \text{and} \quad \theta \bullet (x^{p+3}\log(x)) = x^{p+3} + (p+3)x^{p+3}\log(x)\,.$$
Thus, the matrix of the operator θ in the basis $\{x^{p+3}, x^{p+3}\log(x)\}$ is
$$\begin{bmatrix} p+3 & 1 \\ 0 & p+3 \end{bmatrix}.$$
7. Let $c_{p,1}$ and $c_{p,2}$ be the coefficients of x^{p+3} and $x^{p+3}\log(x)$ in the power series expansion. Then we can write our operators as matrices, and our **recurrence** as
$$\begin{bmatrix} p & 1 \\ 0 & p \end{bmatrix} \begin{bmatrix} p+3 & 1 \\ 0 & p+3 \end{bmatrix} \begin{bmatrix} c_{p,1} \\ c_{p,2} \end{bmatrix} = \begin{bmatrix} p-a+2 & 1 \\ 0 & p-a+2 \end{bmatrix} \begin{bmatrix} p-b+2 & 1 \\ 0 & p-b+2 \end{bmatrix} \begin{bmatrix} c_{p-1,1} \\ c_{p-1,2} \end{bmatrix}$$
with initial values $c_{0,1} = 1,\ c_{0,2} = 0$. Solving the recurrence yields
$$c_{p,1} = 0 \qquad \text{and} \qquad c_{p,2} = \frac{(a+3)_p(b+3)_p}{(1)_p(4)_p}\,.$$