

D-Module Techniques for Solving Differential Equations in the Context of Feynman Integrals

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Aim: exploit algebraic geometry behind Feynman integrals

- ◇ extraction of **properties** of Feynman integrals from their PDEs
- ◇ **algorithmic** computation of **series solutions** of PDEs by algebraic methods
- ◇ **evaluation** of Feynman integrals
- ◇ providing a **dictionary** between algebraic analysis and high energy physics

Definition

The **Weyl algebra** is obtained from the free algebra over \mathbb{C}

$$D := \mathbb{C}[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle$$

by imposing the following relations. All generators commute, except ∂_i and x_i :

$$[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1 \quad \text{for } i = 1, \dots, n.$$

From PDEs to D -ideals and vice versa

- ◇ D gathers linear differential operators with polynomial coefficients

$$P = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha, \beta} x^\alpha \partial^\beta, \quad c_{\alpha, \beta} \in \mathbb{C} \quad \rightsquigarrow \text{PDE: } \boxed{P \bullet f(x_1, \dots, x_n) = 0}$$

Example: $P = \partial^2 - x \in D$ encodes Airy's equation $f''(x) - x \cdot f(x) = 0$.

- ◇ left D -ideals encode systems of linear PDEs
operations with D -ideals: integral transforms, restrictions, push forward, ...

One variable

A function $f(x)$ is **holonomic** if there exists $P \in D$ that annihilates f , i.e., $P \bullet f = 0$.

Multivariate case: $f(x_1, \dots, x_n)$ is holonomic if $\text{Ann}_D(f)$ is a “holonomic” D -ideal.

Examples: Feynman integrals, hypergeometric, periods, Airy, some probability distributions, ...

Denote by $R_n = \mathbb{C}(x_1, \dots, x_n)\langle \partial_1, \dots, \partial_n \rangle$ the **rational Weyl algebra**.

Theorem (Cauchy–Kovalevskaya–Kashiwara)

Let I be a holonomic D -ideal. The \mathbb{C} -vector space of holomorphic solutions to I on a simply connected domain in \mathbb{C}^n outside the **singular locus** of I has finite dimension

$$\text{rank}(I) = \dim_{\mathbb{C}(x_1, \dots, x_n)} (R_n / R_n I) .$$

⇒ A holonomic function is encoded by finite data!

Singularities

D -ideals can be **regular singular** or **irregular singular**.

Univariate case: read from growth behavior of general solution near singular points

Example: $\diamond \log(x)$ **moderate growth** at $x = 0$ $\diamond \exp(1/x)$ **essential singularity** at $x = 0$

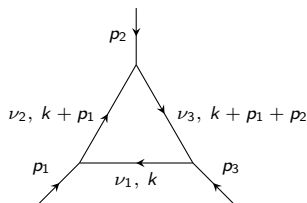
Running example

Variables: $x_1 = |p_1|^2$, $x_2 = |p_2|^2$, $x_3 = |p_1 + p_2|^2$.

The D -ideal $I_3(c_0, c_1, c_2, c_3)$

Consider $I_3(c_0, c_1, c_2, c_3) = \langle P_1, P_2, P_3 \rangle \subset D_3$
arising from **conformal invariance**.

dilatations + conformal boosts



One-loop triangle Feynman diagram with massless propagators and massive external particles.

$$\begin{aligned}
 P_1 &= 4(x_1 \partial_1^2 - x_3 \partial_3^2) + 2(2 + c_0 - 2c_1) \partial_1 - 2(2 + c_0 - 2c_3) \partial_3, \\
 P_2 &= 4(x_2 \partial_2^2 - x_3 \partial_3^2) + 2(2 + c_0 - 2c_2) \partial_2 - 2(2 + c_0 - 2c_3) \partial_3, \\
 P_3 &= (2c_0 - c_1 - c_2 - c_3) + 2(x_3 \partial_3 + x_2 \partial_2 + x_1 \partial_1).
 \end{aligned}$$

Parameters: $c_0 = d$ spacetime dimension c_1, c_2, c_3 conformal weights

Choice: $I_3 := I_3(4, 2, 2, 2) \hat{=} \text{conformal } \phi^4\text{-theory in 4 spacetime dimensions}$

I_3 is regular singular, $\text{rank}(I_3) = 4$

Remark: The D -ideal I_3 is the restriction of a GKZ system.

Principal symbol ($n = 1$)

$\text{in}_{(0,1)}(x\partial - x^2) = x\xi$ is the part of maximal $(0,1)$ -weight $\partial \rightsquigarrow \xi$

Several variables: $\text{in}_{(0,1)}(x_1\partial_1 + x_2\partial_2 + 1) = x_1\xi_1 + x_2\xi_2$ in general, not a monomial

Algebraically

- ◇ The **characteristic ideal** of a D -ideal I is

$$\text{in}_{(0,1)}(I) = \langle \text{in}_{(0,1)}(P) \mid P \in I \rangle \subset \mathbb{C}[x_1, \dots, x_n][\xi_1, \dots, \xi_n].$$

- ◇ The **characteristic variety** of I is

$$\text{Char}(I) = V(\text{in}_{(0,1)}(I)) = \{(x, \xi) \mid p(x, \xi) = 0 \text{ for all } p \in \text{in}_{(0,1)}(I)\} \subset \mathbb{C}^{2n}.$$

- ◇ The **singular locus** $\text{Sing}(I)$ of I is the vanishing set of the ideal

$$(\text{in}_{(0,1)}(I) : \langle \xi_1, \dots, \xi_n \rangle^{(\infty)}) \cap \mathbb{C}[x_1, \dots, x_n]. \quad \text{saturation + elimination}$$

Examples

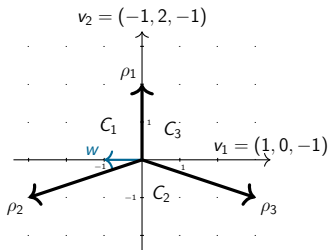
- ① For $I = \langle x^2\partial + 1 \rangle \subset D$, $\text{in}_{(0,1)}(I) = \langle x^2\xi \rangle$ and $\text{Sing}(I) = V(x) = \{0\}$. $\mathbb{C} \cdot \exp(1/x)$
- ② The characteristic ideal of $I = \langle x_1\partial_2, x_2\partial_1 \rangle \subset D_2$ is the $\mathbb{C}[x_1, x_2, \xi_1, \xi_2]$ -ideal $\langle x_1\xi_2, x_2\xi_1, x_1\xi_1 - x_2\xi_2, x_2\xi_2^2, x_2^2\xi_2 \rangle$ and $\text{Sing}(I) = V(x_1, x_2) \subset \mathbb{C}^2$. $\mathbb{C} \cdot 1$

Weights of the form $(-w, w)$, $w = (w_1, \dots, w_n) \in \mathbb{R}^n$

- ◇ The **w-weight** of $c_{\alpha, \beta} x^\alpha \partial^\beta$ is $-w \cdot \alpha + w \cdot \beta$.
- ◇ The **initial form** of $P = \sum c_{\alpha, \beta} x^\alpha \partial^\beta$ is the subsum of all terms of **maximal** w -weight.

Initial and indicial ideal (with respect to w)

- ◇ The **initial ideal** of I is the D -ideal $\text{in}_w(I) = \langle \text{in}_{(-w, w)}(P) \mid P \in I \rangle \subset D$.
- ◇ The **indicial ideal** of I is the $\mathbb{C}[\theta_1, \dots, \theta_n]$ -ideal $\text{ind}_w(I) = R_n \cdot \text{in}_{(-w, w)}(I) \cap \mathbb{C}[\theta_1, \dots, \theta_n]$. $\theta_i = x_i \partial_i$ the i -th Euler operator



Small Gröbner fan of $I_3 \subset D$, here drawn in $\mathbb{R}_w^3 / \mathbb{R}(1, 1, 1)$.

The zeroes of $\text{ind}_w(I)$ in \mathbb{C}^n are the **exponents** of I .

The starting monomials of solutions to I will be of the form $x^A \log(x)^B$ with $A \in V(\text{ind}_w(I))$.

Pipeline: from I to starting terms of series solutions

$$D_n\text{-ideal } I \xrightarrow{w \in \mathbb{R}^n} \text{in}_{(-w, w)}(I) \rightsquigarrow \text{ind}_w(I) \subset \mathbb{C}[\theta_1, \dots, \theta_n] \xrightarrow{V(\text{ind}_w(I))} x^A \log(x)^B$$

Aim: Solutions to I of the form $F_k(x) = x^A \cdot \sum_{\substack{0 \leq p \cdot w \leq k, p \in \mathbb{C}_{\mathbb{Z}}^*, \\ 0 \leq b_j < \text{rank}(I)}} c_{pb} x^p \log(x)^b$.

Initial series

The **w-weight** of a monomial $x^A \log(x)^B$ is the real part of $w \cdot A$. The **initial series** in $w(f)$ of a function $f = \sum_{A,B} c_{AB} x^A \log(x)^B$ is the subsum of all terms of **minimal** w -weight.

Proposition

If I is regular holonomic and w a generic weight for I , there exist $\text{rank}(I)$ many canonical series solutions of I which lie in the **Nilsson ring** $N_w(I)$ of I with respect to w ,

$$N_w(I) := \mathbb{C}[[C_w(I)_{\mathbb{Z}}^*]][x^{e^1}, \dots, x^{e^r}, \log(x_1), \dots, \log(x_n)].$$

- ◇ $C_w(I)^*$ the dual cone of the Gröbner cone of w
- ◇ $C_w(I)_{\mathbb{Z}}^* = C_w(I)^* \cap \mathbb{Z}^n$
- ◇ $\{e^1, \dots, e^r\}$ the exponents of I

Monomial ordering \prec_w refining w -weight: The number of solutions to I with starting monomial of the form $x^A \log(x)^B$ is the multiplicity of A as zero of $\text{ind}_w(I)$.

Theorem (Saito–Sturmfels–Takayama)

Let I be a regular holonomic $\mathbb{Q}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ -ideal and $w \in \mathbb{R}^n$ generic for I . Let I be given by a Gröbner basis for w . There exists an algorithm which computes all terms up to specified w -weight in the canonical series solutions to I with respect to \prec_w .

Procedure

Input: A regular holonomic D_n -ideal I , its small Gröbner fan Σ in \mathbb{R}^n , a weight vector $w \in \mathbb{R}^n$ that is generic for I , and the desired order $k \in \mathbb{N}$.

... for each starting monomial $x^A \log(x)^B$: solving linear system modulo desired w -weight for vector spaces of monomials of same w -weight. [recurrence relations](#)

Output: The canonical series solutions of I with respect to w , truncated at w -weight k .

Starting monomials for I_3

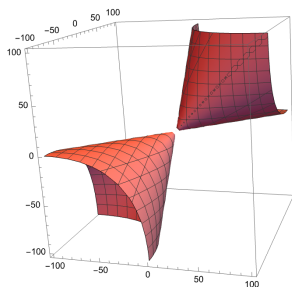
The **singular locus** of I_3 is

$$\text{Sing}(I_3) = V(x_1 x_2 x_3 \cdot \lambda) \subset \mathbb{C}^3.$$

Vanishing locus of the Källén polynomial

$$\lambda = x_1^2 + x_2^2 + x_3^2 - 2(x_1 x_2 + x_1 x_3 + x_2 x_3)$$

+ coordinate hyperplanes $\{x_i = 0\}$



Initial and indicial ideal for $w = (-1, 0, 1) \in C_1$

$$\diamond \text{in}_{(-w, w)}(I_3) = \langle x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + 1, x_2 \partial_2^2 + \partial_2, x_3 \partial_3^2 + \partial_3 \rangle \subset D_3$$

$$\diamond \text{ind}_w(I_3) = R_3 \cdot \text{in}_{(-w, w)}(I) \cap \mathbb{C}[\theta_1, \theta_2, \theta_3] = \langle \theta_1 + \theta_2 + \theta_3 + 1, \theta_2^2, \theta_3^2 \rangle \subset \mathbb{C}[\theta_1, \theta_2, \theta_3]$$

Exponents of I : $V(\text{ind}_w(I_3)) = \{(-1, 0, 0)\} \hat{=} x_1^{-1} x_2^0 x_3^0 = 1/x_1$

Change of variables: $y_1 = x_1, \quad y_2 = x_2/x_1, \quad y_3 = x_3/x_1.$

Starting monomials of solutions *read from primary decomposition of $\text{ind}_w(I)$*

$$\diamond 1/y_1 \quad \diamond 1/y_1 \log(y_2) \quad \diamond 1/y_1 \log(y_3) \quad \diamond 1/y_1 \log(y_2) \log(y_3)$$

Canonical series solutions of I_3

Lifting the starting monomials *here displayed for f_1, f_2, f_3 for w -weight 0 to 4*

$$\begin{aligned}\tilde{f}_1(y_2, y_3) &= 1 + y_2 + y_3 + y_2^2 + 4y_2y_3 + y_3^2 + y_2^3 + 9y_2^2y_3 + y_2^4 + \dots, \\ \tilde{f}_2(y_2, y_3) &= \log(y_2) + \log(y_2)y_2 + (2 + \log(y_2))y_3 + \log(y_2)y_2^2 + (4 + 4\log(y_2))y_2y_3 \\ &\quad + (3 + \log(y_2))y_3^2 + (\log(y_2))y_2^3 + (6 + 9\log(y_2))y_2^2y_3 + \log(y_2)y_2^4 + \dots, \\ \tilde{f}_3(y_2, y_3) &= \log(y_3) + (2 + \log(y_3))y_2 + \log(y_3)y_3 + (3 + \log(y_3))y_2^2 \\ &\quad + (4 + 4\log(y_3))y_2y_3 + \log(y_3)y_3^2 + \left(\frac{11}{3} + \log(y_3)\right)y_2^3 \\ &\quad + (15 + 9\log(y_3))y_2^2y_3 + \left(\frac{25}{6} + \log(y_3)\right)y_2^4 + \dots.\end{aligned}$$

Then $f_i(x_1, x_2, x_3) = 1/x_1 \cdot \tilde{f}_i(y_2, y_3)$ are canonical series solutions to I_3 . (truncated)

Code

- ◇ in Sage for the bivariate case: <https://mathrepo.mis.mpg.de/DModulesFeynman/>
- ◇ Macaulay2 package HolonomicSystems M. Sayrafi, C. Berkesch, A. Leykin, H. Tsai

Truncation with respect to w -weight

$f(x_1, \dots, x_n)$ general solution of a regular holonomic D -ideal I

Capturing the weight vector via an auxiliary variable

Choose a generic weight $w \in \mathbb{R}^n$ for I . Set

$$f_w(t, x_1, \dots, x_n) := f(t^{w_1} x_1, \dots, t^{w_n} x_n).$$

Merging with canonical series solutions

- 1 From I , derive a **Fuchsian system** for $f_w(t, x_1, \dots, x_n)$.
- 2 Solve the system via the path-ordered exponential formalism.
- 3 Compute the asymptotic expansion of $f_w(t, x)$ around $t = 0$:

$$f_w(t, x) = \sum_{k \geq 0} \sum_{m=0}^{m_{\max}} c_{k,m}(x) t^k \log(t)^m.$$

By construction, $c_{k,m}(x)$ has w -weight k .

- 4 Truncate the expansion at t^k and evaluate at $t = 1$. **Nota bene:** $f_w|_{t=1} \equiv f$.

1. F. Brown. Iterated Integrals in Quantum Field Theory. In *6th Summer School on Geometric and Topological Methods for Quantum Field Theory*, pages 188–240, 2013.

2. W. Wasow. *Asymptotic expansions for ordinary differential equations*. Pure and Applied Mathematics, Vol. XIV. Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1965.

In a nutshell

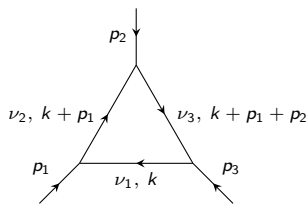
- ① D -ideals encode crucial properties of their solution functions
e.g. Feynman integrals, arbitrary loop order, irrespective of whether polylogarithmic, etc.
- ② algorithmic computation of truncated series solutions by algebraic methods
no gauge transform required
- ③ evaluation of solution functions to desired w -weight
freedom in choosing weight vector w
- ④ dictionary algebra–physics
computing series solutions, Pfaffian system vs. Laporta’s algorithm

Thank you for your attention!

The solution space of I_3 ...

... is spanned by the triangle integral

$$J_{d;\nu_1,\nu_2,\nu_3}^{\text{triangle}} = \int_{\mathbb{R}^d} \frac{1}{(-|k|^2)^{\nu_1} (-|k+p_1|^2)^{\nu_2} (-|k+p_1+p_2|^2)^{\nu_3}} \frac{d^d k}{i\pi^{\frac{d}{2}}}$$

and its analytic continuations. $\text{rank}(I_3) = 4$ 

One-loop triangle Feynman diagram with massless propagators and massive external particles.

Unitary exponents $\nu_1 = \nu_2 = \nu_3 = 1$, $d = 4$:

$$\begin{aligned} f_1(x_1, x_2, x_3) &= J_{4;1,1,1}^{\text{triangle}}(x_1, x_2, x_3), \\ f_2(x_1, x_2, x_3) &= \frac{1}{\sqrt{\lambda}} \log \left(\frac{x_1 - x_2 - x_3 - \sqrt{\lambda}}{x_1 - x_2 - x_3 + \sqrt{\lambda}} \right), \\ f_3(x_1, x_2, x_3) &= \frac{1}{\sqrt{\lambda}} \log \left(\frac{x_2 - x_1 - x_3 - \sqrt{\lambda}}{x_2 - x_1 - x_3 + \sqrt{\lambda}} \right), \\ f_4(x_1, x_2, x_3) &= \frac{1}{\sqrt{\lambda}}, \end{aligned}$$

where $\lambda = x_1^2 + x_2^2 + x_3^2 - 2(x_1x_2 + x_1x_3 + x_2x_3)$ is the **Källén** function.

The conformal group

$z = (z^0, z^1, \dots, z^{d-1})^\top$ vector of d -dimensional spacetime coordinates
 $z_1 \cdot z_2 := z_1^\top \cdot g \cdot z_2$ $g = \text{diag}(1, -1, \dots, -1)$ the metric tensor
 p_1, \dots, p_n momentum vectors

Translations	$z \rightarrow z + \epsilon, \quad \epsilon \in \mathbb{R}^d$
(Proper) Lorentz transformations	$z \rightarrow \Lambda \cdot z, \quad \Lambda \in \text{SO}(1, d-1)$
Dilatations	$z \rightarrow e^\omega z, \quad \omega \in \mathbb{R}$
Conformal boosts	$z \rightarrow \frac{z - z ^2 \epsilon}{1 - 2z \cdot \epsilon + z ^2 \epsilon ^2}, \quad \epsilon \in \mathbb{R}^d$

Poincaré group symmetry group of Einstein's theory of special relativity
conformal group Poincaré + dilatations + conformal boosts

Invariance under...

- ◊ translations implies momentum conservation
- ◊ Lorentz transformation implies dependency on Mandelstam invariants $p_k \cdot p_\ell$ only

Generators in position space to momentum space via Fourier transform

- ◊ dilatations: $\mathcal{D}_n = -i \sum_{k=1}^n (z_k \cdot \partial_{z_k} + c_k)$
- ◊ conformal boosts: $\mathfrak{K}_n = i \sum_{k=1}^n [|z_k|^2 \partial_{z_k} - 2 z_k (z_k \cdot \partial_{z_k}) - 2 c_k z_k]$

Running example: $n = 3$, momenta p_1, p_2, p_3 , variables $x_i = |p_i|^2$

- ◊ P_3 stems from $\widehat{\mathcal{D}}_3$
- ◊ P_1, P_2 stem from $\widehat{\mathfrak{K}}_3$

Systems in matrix form

- ◇ I a holonomic D_n -ideal of rank $m = \text{rank}(I)$, $f \in \text{Sol}(I)$
- ◇ $1, s_2, \dots, s_m$ a $\mathbb{C}(x)$ -basis of $R_n/R_n I$ standard monomials for a Gröbner basis of I

Pfaffian system

Set $F = (f, s_2 \bullet f, \dots, s_m \bullet f)^\top$. There exist $P_1, \dots, P_n \in \mathbb{C}(x_1, \dots, x_n)^{m \times m}$ for which

$$\partial_i \bullet F = P_i \cdot F.$$

The matrices P_i fulfill $P_i P_j - P_j P_i = \partial_i \bullet P_j - \partial_j \bullet P_i$ for all i, j . integrability

If all poles are of order at most 1, the system is **Fuchsian**. To arrive at a Fuchsian form, one might need a gauge transform. Wasow's method

Construction of a Pfaffian system IBP reduction with Laporta's algorithm

∂^a

a in ∂^a

$\partial^a Q_i = 0$ in $R_n/R_n I$

$\mathbb{C}(x)$ -basis of $R_n/R_n I$

Feynman integrals

propagator powers

IBP identities

set of master integrals

1. V. Chestnov, F. Gasparotto, M. K. Mandal, P. Mastrolia, S.-J. Matsubara-Heo, H. J. Munch, N. Takayama. Macaulay matrix for Feynman integrals: Linear relations and intersection numbers. *J. High Energy Phys.*, 187(2022), 2022.

2. W. Wasow. *Asymptotic expansions for ordinary differential equations*. Pure and Applied Mathematics, Vol. XIV. Interscience Publishers John Wiley & Sons, Inc., New York-London- Sydney, 1965.

The SST algorithm

Input: A regular holonomic D_n -ideal I , its small Gröbner fan Σ in \mathbb{R}^n , a weight vector $w \in \mathbb{R}^n$ that is generic for I , and the desired order $k+1 \in \mathbb{N}$.

- 1 Determine a Gröbner basis $G = \{g_1, \dots, g_d\}$ of I with respect to w .
- 2 Write each $g \in G$ as $x^\alpha g = f - h$ with $\alpha \in \mathbb{Z}^n$ such that $f \in \mathbb{K}[\theta_1, \dots, \theta_n]$ and $h \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \langle \partial_1, \dots, \partial_n \rangle$ with $\text{ord}_{(-w, w)}(h) < 0$.
- 3 Compute the indicial ideal $\text{ind}_w(I)$ and its $\text{rank}(I)$ many solutions. They are the form $x^A \log(x)^B$ with $A \in V(\text{ind}_w(I))$. For each starting of these monomials, carry out Step 4.
- 4 Assume the partial solution

$$F_k(x) = x^A \cdot \sum_{0 \leq p \cdot w \leq k, p \in C_{\mathbb{Z}}^*} c_{pb} x^p \log(x)^b.$$

is known. Solve the linear system

$$(f_1, \dots, f_d) \bullet E_{k+1}(x) = (h_1 - f_1, \dots, h_d - f_d) \bullet F_k(x) \text{ mod } w\text{-weight } k+2$$

for $E_{k+1} \in \sum_{p \cdot w = k+1, p \in C_{\mathbb{Z}}^*} L'_p$ of w -weight $k+1$. Adding E_{k+1} to F_k lifts F_k to F_{k+1} .
 L'_p the subspace of $L_p = x^A \sum_{0 \leq b_i \leq \text{rank}(I)} \mathbb{K} \cdot x^p \log(x)^b$ spanned by monomials $\notin \text{Start}_{\prec_w}(I)$

Output: The canonical series solutions of I with respect to w , truncated at w -weight $k+1$.

SST algorithm: a hypergeometric example

Consider the D -ideal I generated by $P = \theta(\theta - 3) - x(\theta + a)(\theta + b)$.

- 1 I is holonomic of rank $\text{ord}_{(0,1)}(P) = 2$.
- 2 Gröbner fan of I : two maximal cones $\pm \mathbb{R}_{\geq 0}$.
- 3 For the weight $w = 1$, $\text{in}_{(-w,w)}(I) = \langle \theta(\theta - 3) \rangle = \text{ind}_w(I)$.
- 4 Exponents of I : $V(\text{ind}_w(I)) = \{0, 3\}$. starting monomials $x^0 = 1$ and x^3
- 5 Choose x^3 as starting monomial, $L_p = \mathbb{C} \cdot \{x^{p+3}, x^{p+3} \log(x)\}$. $x^3 \sum_p c_{p,1} x^p + c_{p,2} x^p \log(x)$
- 6 Write $P = f - h$, where $f = \theta(\theta - 3)$ and $h = x(\theta + a)(\theta + b)$. Action of θ on L_p :

$$\theta \bullet x^{p+3} = (p+3)x^{p+3} \quad \text{and} \quad \theta \bullet (x^{p+3} \log(x)) = x^{p+3} + (p+3)x^{p+3} \log(x).$$

Thus, the matrix of the operator θ in the basis $\{x^{p+3}, x^{p+3} \log(x)\}$ is

$$\begin{bmatrix} p+3 & 1 \\ 0 & p+3 \end{bmatrix}.$$

- 7 Let $c_{p,1}$ and $c_{p,2}$ be the coefficients of x^{p+3} and $x^{p+3} \log(x)$ in the power series expansion. Then we can write our operators as matrices, and our **recurrence** as

$$\begin{bmatrix} p & 1 \\ 0 & p \end{bmatrix} \begin{bmatrix} p+3 & 1 \\ 0 & p+3 \end{bmatrix} \begin{bmatrix} c_{p,1} \\ c_{p,2} \end{bmatrix} = \begin{bmatrix} p-a+2 & 1 \\ 0 & p-a+2 \end{bmatrix} \begin{bmatrix} p-b+2 & 1 \\ 0 & p-b+2 \end{bmatrix} \begin{bmatrix} c_{p-1,1} \\ c_{p-1,2} \end{bmatrix}$$

with initial values $c_{0,1} = 1, c_{0,2} = 0$. Solving the recurrence yields

$$c_{p,1} = 0 \quad \text{and} \quad c_{p,2} = \frac{(a+3)_p (b+3)_p}{(1)_p (4)_p}.$$