YOGA WITH POLYLOGARITHMS

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Warm-up for the talks

today: $g=0,1$

- Iterated integrals on Riemann surfaces of different genus — Thu, Feb 15
  Oliver Schlotterer (Uppsala Universitet)
- My favorite elliptic hypergeometric integral — Fr, Feb 16
  Carlos Rodriguez (Uppsala Universitet)

References:

- F. Brown, A. Levin: **Multiple Elliptic Polylogrithms.** arXiv: 1110.6917
- J. Broedel, O. Schlotterer, S. Stieberger: **Polylogarithms, multiple zeta values, and superstring amplitudes.** arXiv: 1304.7267
- J. Broedel, C. Duhr, F. Dulat, L. Tancredi: **Elliptic polylogs and iterated integrals on elliptic curves I.** arXiv: 1712.07089
- J. Broedel, C. Duhr, F. Dulat, B. Penante, L. Tancredi: **Elliptic polylogarithms and Feynman parameter integrals.** arXiv: 1902.09971

1. POLYLOGARITHMS not closed under $\int$

→ date back to L. Euler

$s \in \mathbb{C}$

$$\mathrm{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s} = x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \dots$$

conv. for $x \in \mathbb{C}$ with $|x| < 1$
$\rightsquigarrow$ extend to $|x| \geq 1$ by analytic continuation
branch points at 0 and 1
$\rightsquigarrow$ monodromy

multivalued fct. on $\mathbb{C} \setminus \{0,1\}$
for $s \notin \mathbb{Z}$: requires choice of branch ($e^{s \cdot \ln(k)}$)

Polylogs as iterated integrals:

$$\mathrm{Li}_{s+1}(x) = \int_0^x \frac{\mathrm{Li}_s(t)}{t}\, dt$$

$$\Rightarrow \mathrm{Li}_n(x) = \int_{0 \leq t_1 \leq \dots \leq t_n \leq x} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \cdots \frac{dt_n}{t_n}$$

↑ integration kernels: at most simple poles
⇒ polylogs have log. sing. at most

Some special cases:

$s=1$: $\mathrm{Li}_1(x) = -\ln(1-x) = \int_0^x \frac{1}{1-t}\, dt$

$s=2$: $\mathrm{Li}_2(x) = \int_{0 \leq t_1 \leq t_2 \leq x} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2}$ dilogarithm

$s=3$: trilogarithm

$s \in \mathbb{Z}_{\leq 0}$: $\mathrm{Li}_s \in \mathbb{C}(x)$

Differential and difference equations for polylogs

$\underbrace{x\partial_x}_{=\theta_x} \bullet \mathrm{Li}_s = \mathrm{Li}_{s-1}$ differential - difference equation

Euler operator

integer shifts of s

Annihilating differential operators ($\in D$)

$n=1$: $(x-1)\partial^2 + \partial$

$n=2$ $(x-1)\,x\partial^3 + (3x-2)\partial^2 + \partial$

$n=3$: $(x-1)\,x^2\partial^4 + (6x-5)x\,\partial^3 + (7-4x)\partial^2 + \partial$

$\vdots$

$\cdot\,\theta$ by the right
$(P\bullet \mathrm{Li}_s = 0 \Rightarrow (P\theta)\cdot \mathrm{Li}_{s+1} = 0)$

Mathematica package Holonomic functions by Christoph Koutschan

relevant integer sequences (OEIS):
A212342 (conj.)
A000225 (# rank-1 matroids over S_n)
...

Relation to some special functions:

- Riemann ζ-fct $\mathrm{Li}_s(1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ $(\mathrm{Re}(s)>1)$ analytic contin. elsewhere
- generalized hypogeom. fct: ${}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{x^k}{k!}$

$\mathrm{Li}_n(x) = x \cdot {}_{n+1}F_n(1,1,\dots,1;2,\dots,2;x)$ $n=0,1,2,\dots$

$\mathrm{Li}_{-n}(x) = x\;{}_nF_{n-1}(2,\dots,2;1,\dots,1;x)$ $n=1,2,3,\dots$

2. MULTIPLE POLYLOGARITHMS

→ sufficient to express large class of Feynman integrals!

sometimes called "hyperlogarithm"

$\in \mathbb{C}$, poles of rational integration kernels

$$\underset{\text{Goncharov}}{G}(\underbrace{a_1, \dots, a_n}_{\text{"labels" } \vec{a}}; \underset{\text{"argument"}}{x}) = \int_0^x \frac{G(a_2, \dots, a_n; t)}{t - a_1} \, dt$$

"weight" n

integrating a rat. fct on Riemann surface of genus 0

pole order 1! ⇒ MPLs have logarithmic singularities!

- $G(x) = G(;x) = 1$ except for $x = 0$: $G(\vec{a};0) = G(;0) = 0$
- $\vec{a} = 0$: $G(\vec{a};x)$ divergent; $G(0, \dots, 0; x) := \frac{1}{n!}(\log x)^n$, $\log x = \int_1^x \frac{dt}{t}$

Polylogs as special case:

For $\vec{a} = (\underbrace{0, \dots, 0}_{n-1 \text{ many}}, a_n)$: $\boxed{G(\vec{a};x) = -\mathrm{Li}_n(x/a_n)}$

Structure: $\{\text{MPL}\}$ closed under $\int$!

MPLs form a graded (by the weight) Lie algebra, with shuffle product:

$$G(a_1, \dots, a_r; x)\, G(a_{r+1}, \dots, a_{r+s}; x) = \sum_{\sigma \in \Sigma(r,s)} G(a_{\sigma(1)}, \dots, a_{\sigma(r+s)}; x).$$

↳ next slide

Neutral element for shuffling: $G(;z) = 1$

$\Sigma(r,s) \subset S_{r+s}$: subgroup of permutations which leave the orders of $\{a_1,\dots,a_r\}$ and $\{a_{r+1},\dots,a_{r+s}\}$ unchanged

Example $(r=3,\ s=2)$

$\Sigma(3,2) \subset S_5$

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_4 & a_1 & a_2 & a_5 & a_3 \end{pmatrix} \checkmark \qquad \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_5 & a_3 & a_4 & a_1 \end{pmatrix} \times$$

Shuffle product:

$\vec{a} \sqcup\!\sqcup \vec{b}$ sum over all permutations of $\vec{a} \cup \vec{b}$ that preserve the relative orderings within $\vec{a}$ and $\vec{b}$.

Differentiating MPL's:

skip a_i

$$\underset{\text{w.r.t. } a_i\text{'s}}{dG(a_1,\dots,a_n;x)} = \sum_{i=1}^{n} G(a_1,\dots,\hat{a}_i,\dots,a_n;x)\ d\log\left(\frac{a_{i-1}-a_i}{a_{i+1}-a_i}\right)$$

Multiple zeta values: $\mathrm{MPL}\big|_{z=1}$ **Aim:** relations of MZVs over $\mathbb{Q}$?

3. ELLIPTIC POLYLOGARITHMS (eMPLs)

$\{MPL\} \subset \{eMPL\}$

→ D. Zagier, Brown-Levin

MATHEMATICAL MOTIVATION

- Iterated integrals on $\mathcal{M}_{0,n}$: can be expressed in terms of MPLs
- iterated integrals on $\mathcal{E}^{(n)}$: can be expressed in terms of eMPLs

  configuration space of n+1 marked points on ell. curve $\mathcal{E}$

Physics: special case ("essentially same fct. space")

→ useful for one-loop scattering amplitudes in open superstring theory

WHAT?

Iterated integrals, integration kernels: rational functions on ell. curve

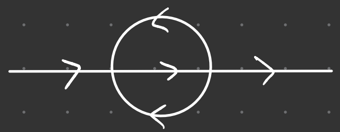
$R \in \mathbb{C}(x,y) / \langle y^2 - P_3(x) \rangle$

ell. curve $\mathcal{E} = \{ y^2 = P_3(x) \}$

WHY?

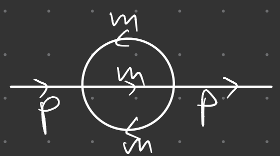
- properties similar to MPLs, closed under $\int$, shuffle algebra ⚠ but higher poles allowed
- occur as coefficients in Laurent expansion in "small parameter" ε, e.g. for two-loop massive sunrise graph

  3 generic internal masses



Example (Lecture by Claude Duhr: Polylogs, ell. polylogs, etc. → YouTube
Summer School 2021, Mainz: "The Amplitudes Games")

2-loop sunrise diagram
w. equal masses



$$I_G = \int \frac{d^2k \, d\ell^2}{(k^2-m^2)(\ell^2-m^2)((k+\ell)^2-m^2)}$$

Feynman integral of G

loop momentum

$$= -\int_{\mathbb{R}^3_{\geq 0}} \frac{dx_1\, dx_2\, dx_3\, \delta(1-x_3)}{F(x_1, x_2, x_3)}$$

putting $x_3 = 1$

2nd Symanzik polynomial

$$F = (-p^2)\, x_1 x_2 x_3 + m^2 (x_1+x_2+x_3)(x_1x_2 + x_1x_3 + x_2x_3)$$

do integral in x_2

With $x_1 = x/_{1-x}$, the integral becomes

"differential 1st kind"

$$\frac{1}{m^2-p^2} \cdot \int_0^1 \frac{dx}{y} \underbrace{\log\left(\frac{x(1-x)t + t + (t+1)y}{x(1-x)t + t - (t+1)y}\right)}_{= \log(R(x,y))}$$

$\sqrt{\ }$ in x $\Rightarrow$ not a MPL!

→ eMPL!

$t = m^2/_{(-p)^2}$ kinematic variable

$y^2 = P_4(x)$

Can be expressed in terms of eMPLs! I_G is an elliptic dilogarithm.

INTEGRATING OVER ELLIPTIC CURVES

$\mathcal{E} = \{y^2 = P_3(x)\}$ ell. curve

$p_i, q_i \in \mathbb{C}[x]$, $R_i \in \mathbb{C}(x)$

$R \in \mathbb{C}(x,y) / \langle y^2 - P_3(x) \rangle$

$$R(x,y) = \frac{p_1(x) + p_2(x)\sqrt{P_3}}{q_1(x) + q_2(x)\sqrt{P_3}} = R_1(x) + \frac{1}{y} R_2(x)$$

$$\Rightarrow \boxed{\int R(x,y)\,dx = \int R_1(x)\,dx + \int R_2(x)\,\frac{dx}{y}}$$

rational fct's + log

↪ partial fraction decomposition: $\frac{x\,dx}{y}$, $\frac{dx}{y(x-c)^m}$

Required integration kernels for eMPLs:

differential of the...

$\frac{dx}{y}$	$\frac{x\,dx}{y}$	$\frac{dx}{y(x-c)}$	$\frac{dx}{x-c}$
1st kind	2nd kind	3rd kind	
no pole	double pole at ∞, zero residue	simple poles at $x=c$	at $x=c,\infty$

PARAMETERIZING ELLIPTIC CURVES

via torus + Weierstraß $\wp$-function ← script p

$\mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\omega_1, \omega_2 \in \mathbb{C}$, lin. independent$|_{\mathbb{R}}$

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

elliptic function (double-periodic merom. fct.)

After scaling: $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, $\mathrm{Im}(\tau) > 0$

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right)$$

Differential equation for $\wp$: $\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3$ $\quad g_2, g_3 \in \mathbb{C}$

not holonomic, but D-algebraic!

The image of

$$\mathbb{C}/\Lambda \longrightarrow \mathbb{C}^2, \quad [z] \longmapsto (x,y) = (\wp(z), \wp'(z))$$

$z \in \Lambda \longmapsto x = \infty$

is an elliptic curve.

For every elliptic curve, $\exists$ coordinate change s.t. $\mathcal{E} = \{y^2 = 4x^3 - g_2 x - g_3\}$

Differential forms: $\dfrac{dx}{y} = \dfrac{d\wp}{\wp'} = \dfrac{\wp'(z)}{\wp'(z)}\,dz = dz$, $\quad \dfrac{x\,dx}{y} = \left(\dfrac{1}{z^2} + \dfrac{0}{z} + \mathcal{O}(z)\right)dz$

double pole, zero residue

Laurent expansion in small parameter [Section 8.2, 1712.07089]

$n_i, \alpha_i \in \mathbb{Z}$

$$T(n_1, n_2, n_3; x) = \int_0^1 t^{-1/2 + n_1 + \alpha_1 \varepsilon} \cdot (1-x)^{-1/2 + n_2 + \alpha_2 \cdot \varepsilon} \cdot (1 - xt)^{-1/2 + n_3 + \alpha_3 \cdot \varepsilon} \, dt$$

tightly connected to Gauß' hypergeom. fct. ${}_2F_1$

Aim: Laurent expansion in "small parameter" ε

- If exponents are $\in \mathbb{Z}$ for $\varepsilon = 0$: Laurent coefficients are MPLs!
- In general: coefficients expressible in terms of elliptic polylogs!

Sneak preview:

- Carlos: elliptic hypergeometric integrals
- Oli: $g \geq 2$ secret longterm goal: